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The geometry of stable minimal surfaces in metric Lie groups

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ABSTRACT. We study geometric properties of compact stable minimal surfaces with boundary in homogeneous 3-manifolds X that can be expressed as a semidirect product of \mathbb{R}^2 with \mathbb{R} endowed with a left invariant metric. For any such compact minimal surface M , we provide a priori radius estimate which depends only on the maximum distance of points of the boundary ∂M to a vertical geodesic of X . We also give a generalization of the classical Rado's Theorem [30] in \mathbb{R}^3 to the context of compact minimal surfaces with graphical boundary over a convex horizontal domain in X , and we study the geometry, existence and uniqueness of this type of Plateau problem.

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1. Introduction.

In this paper we study the geometry of compact minimal surfaces with boundary in homogeneous manifolds diffeomorphic to \mathbb{R}^3 . By classification, each such homogeneous manifold X is a *metric Lie group*, i.e., a simply connected 3-dimensional Lie group equipped with a left invariant Riemannian metric $\langle \cdot, \cdot \rangle$. For such an X there are two possibilities; either X is isometric to the universal cover of the special linear group $\mathrm{SL}(2, \mathbb{R})$ endowed with some left invariant metric, or X is a *metric semidirect product*. By definition, a metric semidirect product $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ is given as a Lie group $(\mathbb{R}^3 \equiv \mathbb{R}^2 \times \mathbb{R}, *)$ together with a certain left invariant metric (its *canonical metric*, see Definition 2.1), where the product operation $*$ is expressed in terms of some real 2×2 matrix $A \in \mathcal{M}_2(\mathbb{R})$ as

$$(\mathbf{p}_1, z_1) * (\mathbf{p}_2, z_2) = (\mathbf{p}_1 + e^{z_1 A} \mathbf{p}_2, z_1 + z_2);$$

see Subsection 2.1 for more details. When $\mathrm{trace}(A) = 0$, X is a *unimodular* semidirect product; typical examples of Riemannian manifolds in this situation are the Euclidean space \mathbb{R}^3 , the Heisenberg space Nil_3 or the solvable Lie group Sol_3 with its usual Thurston geometry. When $\mathrm{trace}(A) \neq 0$, we obtain the *non-unimodular* semidirect products, among which we highlight the hyperbolic space \mathbb{H}^3 and the Riemannian product $\mathbb{H}^2 \times \mathbb{R}$.

The geometry of minimal surfaces in homogeneous 3-manifolds of non-constant sectional curvature has been deeply studied in the last decade, specially in the case that the isometry group of the homogeneous manifold has dimension four. To indicate just a few relevant works in this area, we may cite [1, 2, 3, 4, 5, 6, 11, 12, 17, 26, 31, 33, 35]. An outline of the beginning of the theory of constant mean curvature surfaces in homogeneous 3-manifolds with a 4-dimensional isometry group can be consulted in [7, 13].

For the generic, *non-symmetric* case of homogeneous 3-manifolds with an isometry group of dimension three, the theory of minimal surfaces is less developed. For some works dealing with this more general situation, see e.g., [8, 9, 10, 14, 15, 16, 19, 20, 21, 24, 27, 29]. For an introduction to the geometry of general simply connected homogeneous 3-manifolds, see [21].

In this paper we develop some aspects of the theory of compact minimal surfaces with boundary in metric semidirect products $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$. We shall be specially interested in the geometry,

existence and uniqueness of solutions to the Plateau problem for graphical boundaries on convex domains of $\mathbb{R}^2 \rtimes_A \{0\}$, and on estimating the radius of compact stable minimal surfaces with boundary in metric semidirect products. We recall that the *radius* of a compact Riemannian surface M with boundary is the maximum distance of points in the surface to its boundary ∂M .

An important classical result of Rado [30] states that a simple closed curve Γ in \mathbb{R}^3 that has a 1-1 orthogonal projection to a convex curve in a plane $P \subset \mathbb{R}^3$, is the boundary of a minimal disk of finite area that is a graph over its projection to P , and furthermore, any branched minimal disk in \mathbb{R}^3 with boundary Γ has similar properties. Other classical Rado type results for minimal surfaces in \mathbb{R}^3 were obtained in [18, 28]. In Section 3 of this paper we will extend Rado's theorem to the context of minimal surfaces in metric semidirect products; see Theorems 3.1 and 3.3.

In Section 4, Theorems 3.1 and 3.3 are used to study the geometry of compact minimal surfaces Σ in a non-unimodular semidirect product, such that Σ is the boundary of a round Euclidean circle in $\mathbb{R}^2 \rtimes_A \{0\}$; namely we prove that as the radius of such a circle goes to infinity, then the angles that Σ makes with $\mathbb{R}^2 \rtimes_A \{0\}$ along its boundary circle converge uniformly to $\pi/2$; see Theorem 4.1 for a generalization of this result and also see the related application given in Corollary 4.3 to the existence of a minimal annulus bounded by two circles in $\mathbb{R}^2 \rtimes_A \{0\}$ of large radius, so that these circles can be taken arbitrarily far away from each other. All of these results are then applied in Section 5 to obtain radius estimates of compact minimal surfaces with boundary in metric semidirect products $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$, as we explain next. Given $A \in \mathcal{M}_2(\mathbb{R})$, any vertical line $\Gamma = \{(x_0, y_0, z) \mid z \in \mathbb{R}\}$ in $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ is a geodesic of X (endowed with its canonical metric), which we call a *vertical geodesic*. By a *metric solid cylinder* of radius $r > 0$ in X around Γ we mean the set of points $\mathcal{W}(\Gamma, r)$ in X whose distance to Γ is at most r . With these definitions in mind, the next theorem summarizes another main result of the paper.

THEOREM 1.1. *Let $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ be a metric semidirect product, and let $\mathcal{W}(\Gamma, r)$ be a solid metric cylinder in X of radius $r > 0$ around a vertical geodesic Γ . There exists some $R = R(r) > 0$ such that if M is a compact, stable minimal surface in X whose boundary ∂M is contained in $\mathcal{W}(\Gamma, r)$, then M has radius at most R . In particular, there are no complete stable minimal surfaces contained in $\mathcal{W}(\Gamma, r)$.*

Theorem 1.1 will be proved in Section 5; see Theorem 5.6. Another tool used to prove Theorem 5.6 is Proposition 5.4, where we will construct a certain family of mean convex solid cylinders over appropriately defined ellipses in non-unimodular metric semidirect products with positive Milnor D -invariant. For this and other purposes, we will prove in the Appendix a few additional technical results about the geometry of these metric semidirect products.

2. Background material on 3-dimensional metric Lie groups.

This preliminary section is devoted to state some basic properties of 3-dimensional Lie groups endowed with a left invariant metric that will be used freely in later sections. For details of these basic properties, see the general reference [21].

Let Y denote a simply connected, homogeneous Riemannian 3-manifold, and assume that it is not isometric to the Riemannian product of the 2-sphere $\mathbb{S}^2(\kappa)$ of constant curvature $\kappa > 0$ with the real line. Then Y is isometric to a simply connected, 3-dimensional Lie group G equipped with a left invariant metric $\langle \cdot, \cdot \rangle$, i.e., for every $p \in G$, the left translation $l_p: G \rightarrow G$, $l_p(q) = pq$, is an isometry of $\langle \cdot, \cdot \rangle$. We will call such a space a *metric Lie group*, $X = (G, \langle \cdot, \cdot \rangle)$. When X is simply connected, there are three possibilities:

- X is isometric to the special unitary group $SU(2)$ with a left invariant metric. This is the only case in which X is not diffeomorphic to \mathbb{R}^3 , and the family of left invariant metrics is 3-dimensional.
- X is isometric to the universal cover $\widetilde{SL}(2, \mathbb{R})$ of the special linear group, equipped with a left invariant metric. Again, there is a 3-dimensional family of such metrics.
- X is isometric to a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ equipped with its *canonical metric*, which is the left invariant metric introduced in Definition 2.1 below. In this third case, the underlying Lie group is $(\mathbb{R}^3 \equiv \mathbb{R}^2 \times \mathbb{R}, *)$, where the group operation $*$ is expressed in terms of some real 2×2 matrix $A \in \mathcal{M}_2(\mathbb{R})$ as

$$(2.1) \quad (\mathbf{p}_1, z_1) * (\mathbf{p}_2, z_2) = (\mathbf{p}_1 + e^{z_1 A} \mathbf{p}_2, z_1 + z_2);$$

here $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{R}^2$, $z_1, z_2 \in \mathbb{R}$ and $e^B = \sum_{k=0}^{\infty} \frac{1}{k!} B^k$ denotes the usual exponentiation of a matrix $B \in \mathcal{M}_2(\mathbb{R})$.

2.1. Semidirect products. Consider the semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$, where

$$(2.2) \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then, in terms of the coordinates $(x, y) \in \mathbb{R}^2$, $z \in \mathbb{R}$, we have the following basis $\{F_1, F_2, F_3\}$ of the linear space of *right invariant* vector fields on $\mathbb{R}^2 \rtimes_A \mathbb{R}$:

$$(2.3) \quad F_1 = \partial_x, \quad F_2 = \partial_y, \quad F_3(x, y, z) = (ax + by)\partial_x + (cx + dy)\partial_y + \partial_z.$$

In the same way, a *left invariant* frame $\{E_1, E_2, E_3\}$ of X is given by

$$(2.4) \quad E_1(x, y, z) = a_{11}(z)\partial_x + a_{21}(z)\partial_y, \quad E_2(x, y, z) = a_{12}(z)\partial_x + a_{22}(z)\partial_y, \quad E_3 = \partial_z,$$

where

$$(2.5) \quad e^{zA} = \begin{pmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{pmatrix}.$$

In terms of A , the Lie bracket relations are:

$$(2.6) \quad [E_1, E_2] = 0, \quad [E_3, E_1] = aE_1 + cE_2, \quad [E_3, E_2] = bE_1 + dE_2.$$

Observe that $\text{Span}\{E_1, E_2\}$ is an integrable 2-dimensional distribution of $\mathbb{R}^2 \rtimes_A \mathbb{R}$, whose integral surfaces are the leaves of the foliation $\mathcal{F} = \{\mathbb{R}^2 \rtimes_A \{z\} \mid z \in \mathbb{R}\}$ of $\mathbb{R}^2 \rtimes_A \mathbb{R}$.

DEFINITION 2.1. We define the *canonical left invariant metric* on the semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ to be that one for which the left invariant basis $\{E_1, E_2, E_3\}$ given by (2.4) is orthonormal. Equivalently, it is the left invariant extension to $\mathbb{R}^2 \rtimes_A \mathbb{R}$ of the inner product on the tangent space $T_{\vec{0}}(\mathbb{R}^2 \rtimes_A \mathbb{R})$ at the identity element $\vec{0} = (0, 0, 0)$ that makes $\{(\partial_x)_{\vec{0}}, (\partial_y)_{\vec{0}}, (\partial_z)_{\vec{0}}\}$ an orthonormal basis.

We next emphasize some other metric properties of the canonical left invariant metric \langle, \rangle on $\mathbb{R}^2 \rtimes_A \mathbb{R}$:

- The mean curvature of each leaf of the foliation $\mathcal{F} = \{\mathbb{R}^2 \rtimes_A \{z\} \mid z \in \mathbb{R}\}$ with respect to the unit normal vector field E_3 is the constant $H = \text{trace}(A)/2$. All the leaves of the foliation \mathcal{F} are intrinsically flat.

- The change from the orthonormal basis $\{E_1, E_2, E_3\}$ to the basis $\{\partial_x, \partial_y, \partial_z\}$ given by (2.4) produces the following expression for the metric \langle, \rangle in the x, y, z coordinates of $X := (\mathbb{R}^2 \rtimes_A \mathbb{R}, \langle, \rangle)$:

$$\begin{aligned}
(2.7) \quad \langle, \rangle &= [a_{11}(-z)^2 + a_{21}(-z)^2] dx^2 + [a_{12}(-z)^2 + a_{22}(-z)^2] dy^2 + dz^2 \\
&+ [a_{11}(-z)a_{12}(-z) + a_{21}(-z)a_{22}(-z)] (dx \otimes dy + dy \otimes dx) \\
&= e^{-2\text{trace}(A)z} \{ [a_{21}(z)^2 + a_{22}(z)^2] dx^2 + [a_{11}(z)^2 + a_{12}(z)^2] dy^2 \} + dz^2 \\
&- e^{-2\text{trace}(A)z} [a_{11}(z)a_{21}(z) + a_{12}(z)a_{22}(z)] (dx \otimes dy + dy \otimes dx).
\end{aligned}$$

- The Levi-Civita connection associated to the canonical left invariant metric is easily deduced from the Koszul formula and (2.6) as follows:

$$(2.8) \quad \begin{array}{l} \nabla_{E_1} E_1 = a E_3 \\ \nabla_{E_2} E_1 = \frac{b+c}{2} E_3 \\ \nabla_{E_3} E_1 = \frac{c-b}{2} E_2 \end{array} \left| \begin{array}{l} \nabla_{E_1} E_2 = \frac{b+c}{2} E_3 \\ \nabla_{E_2} E_2 = d E_3 \\ \nabla_{E_3} E_2 = \frac{b-c}{2} E_1 \end{array} \right| \begin{array}{l} \nabla_{E_1} E_3 = -a E_1 - \frac{b+c}{2} E_2 \\ \nabla_{E_2} E_3 = -\frac{b+c}{2} E_1 - d E_2 \\ \nabla_{E_3} E_3 = 0. \end{array}$$

REMARK 2.2. It follows from equation (2.7) that given $(x_0, y_0) \in \mathbb{R}^2$, the map

$$(x, y, z) \xrightarrow{\phi} (-x + 2x_0, -y + 2y_0, z)$$

is an isometry of $(\mathbb{R}^2 \rtimes_A \mathbb{R}, \langle, \rangle)$ into itself. Note that ϕ is the rotation by angle π around the line $l = \{(x_0, y_0, z) \mid z \in \mathbb{R}\}$, and the fixed point set of ϕ is the geodesic l . In particular, vertical lines in the x, y, z -coordinates of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ are geodesics of its canonical metric, which are the axes or fixed point sets of the isometries corresponding to rotations by angle π around them. For any line L in $\mathbb{R}^2 \rtimes_A \{0\}$, let P_L denote the vertical plane $\{(x, y, z) \mid (x, y, 0) \in L, z \in \mathbb{R}\}$ containing the set of vertical lines passing through L . It follows that the plane P_L is ruled by vertical geodesics and furthermore, since the rotation by angle π around any vertical line in P_L is an isometry that leaves P_L invariant, then P_L has zero mean curvature. Thus, every metric Lie group that can be expressed as a semidirect product of the form $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with its canonical metric has many minimal foliations by parallel vertical planes, where by parallel we mean that the related lines in $\mathbb{R}^2 \rtimes_A \{0\}$ for these planes are parallel in the intrinsic metric.

2.2. Unimodular groups. Among all simply connected, 3-dimensional Lie groups, the cases $\text{SU}(2)$, $\widetilde{\text{SL}}(2, \mathbb{R})$, Sol_3 (whose underlying group arises in the so called *Sol geometry*), $\widetilde{\text{E}}(2)$ (universal cover of the Euclidean group of orientation-preserving rigid motions of the plane), Nil_3 (Heisenberg group) and \mathbb{R}^3 comprise the *unimodular* Lie groups; the cases of Sol_3 , $\widetilde{\text{E}}(2)$, Nil_3 and \mathbb{R}^3 with their left invariant metrics correspond to the metric semidirect products $\mathbb{R}^2 \rtimes_A \mathbb{R}$, where the trace of A is zero. We refer the reader to [21] for further details.

2.3. Non-unimodular groups. The case $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ with $\text{trace}(A) \neq 0$ corresponds to the simply connected, 3-dimensional, *non-unimodular* metric Lie groups. In this case, up to the rescaling of the metric of X , we may assume that $\text{trace}(A) = 2$. *This normalization in the non-unimodular case will be assumed from now on throughout the paper.* After an appropriate orthogonal change of the left invariant frame that fixes the vertical field E_3 , we may express the matrix A uniquely as (see Section 2.5 in [21]):

$$(2.9) \quad A = A(\alpha, \beta) = \begin{pmatrix} 1 + \alpha & -(1 - \alpha)\beta \\ (1 + \alpha)\beta & 1 - \alpha \end{pmatrix}, \quad \alpha, \beta \in [0, \infty).$$

The *canonical basis* of the non-unimodular metric Lie group X is, by definition, the left invariant orthonormal frame $\{E_1, E_2, E_3\}$ given in (2.4) by the matrix A in (2.9). In other words, every simply connected, non-unimodular metric Lie group is isomorphic and isometric (up to possibly rescaling the metric) to $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with its canonical metric, where A is given by (2.9). If $A = I_2$ where I_2 is the identity matrix, we get a metric Lie group that we denote by \mathbb{H}^3 , which is isometric to the hyperbolic 3-space with its standard metric of constant sectional curvature -1 and where the underlying Lie group structure is isomorphic to that of the set of similarities of \mathbb{R}^2 . Under the assumption that $A \neq I_2$, the determinant of A determines uniquely the Lie group structure.

DEFINITION 2.3. The *Milnor D -invariant* of $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ is the determinant of A :

$$(2.10) \quad D = (1 - \alpha^2)(1 + \beta^2) = \det(A).$$

Assuming $A \neq I_2$, given $D \in \mathbb{R}$, one can solve (2.10) for $\alpha = \alpha(D, \beta)$, producing a related matrix $A(D, \beta)$ by equation (2.9), and the space of canonical left invariant metrics on the corresponding non-unimodular Lie group structure is parameterized by the values of $\beta \in [m(D), \infty)$, where

$$(2.11) \quad m(D) = \begin{cases} \sqrt{D-1} & \text{if } D > 1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, after scaling so that $\text{trace}(A) = 2$ and assuming that $A \neq I_2$, the space of simply connected, 3-dimensional, non-unimodular metric Lie groups with a given D -invariant is 1-dimensional.

REMARK 2.4. From now on, by a metric semidirect product X we will mean (without loss of generality, see the explanation below) a semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ endowed with its canonical left invariant metric $\langle \cdot, \cdot \rangle$, and such that the matrix $A \in \mathcal{M}_2(\mathbb{R})$ either has trace zero (unimodular case) or is given by expression (2.9) for some $\alpha, \beta \in [0, \infty)$ (non-unimodular case). We must observe that we do not lose any generality with this normalization, since by the previous discussion, every metric semidirect product $\mathbb{R}^2 \rtimes_B \mathbb{R}$ whose associated matrix B has non-zero trace is both isomorphic and isometric (after an adequate rescaling) to a metric semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ where A is given by (2.9). Moreover, the corresponding isomorphism takes the horizontal foliation $\{\mathbb{R}^2 \rtimes_B \{z\} \mid z \in \mathbb{R}\}$ of $\mathbb{R}^2 \rtimes_B \mathbb{R}$ to the horizontal foliation $\{\mathbb{R}^2 \rtimes_A \{z\} \mid z \in \mathbb{R}\}$ of $\mathbb{R}^2 \rtimes_A \mathbb{R}$ and also preserves the left invariant vertical vector fields E_3 of their respective canonical frames.

Along the paper, we will denote by $\Pi: \mathbb{R}^2 \rtimes_A \mathbb{R} \rightarrow \mathbb{R}^2 \rtimes_A \{0\}$ the projection $\Pi(x, y, z) = (x, y, 0)$.

3. Rado's Theorem in metric semidirect products.

In this section we prove some results concerning the geometry of solutions to Plateau type problems in metric semidirect products $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$, when there is some geometric constraint on the boundary values of the solution. The first of these results is Theorem 3.1 below. We remark that several versions of this theorem in the classical setting of $X = \mathbb{R}^3 = \mathbb{R}^2 \times \mathbb{R}$ were proved by Rado [30], Nitsche [28] and Meeks [18]. We point out that one of the difficulties in obtaining Rado-type results in the situation below is that the vertical translation $(x, y, z) \mapsto (x, y, z + t)$ might not be an isometry of the canonical metric on $\mathbb{R}^2 \rtimes_A \mathbb{R}$.

THEOREM 3.1 (Rado's Theorem in metric semidirect products). *Let $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ be a metric semidirect product. Suppose that E is a compact convex disk in $\mathbb{R}^2 \rtimes_A \{0\}$, $C = \partial E$*

and $\Gamma \subset \Pi^{-1}(C)$ is a continuous simple closed curve such that $\Pi|_{\Gamma}: \Gamma \rightarrow C$ monotonically parameterizes¹ C . Then:

1. Γ is the boundary of a compact embedded disk D of finite least area.
2. The interior of D is a smooth Π -graph over the interior of E .

Theorem 3.1 will be a direct consequence of Theorem 3.3 below, which actually gives a more complete statement. The proof of Theorem 3.3 also depends on the following Lemma 3.2; both of these results will also be used in the proof of Theorem 4.1 in Section 4.

LEMMA 3.2. Suppose $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ is a metric semidirect product. Let $E \subset \mathbb{R}^2 \rtimes_A \{0\}$ be a compact convex disk with boundary curve C . If M is a compact branched minimal surface in X with boundary contained in $\Pi^{-1}(C)$, then:

1. $\text{Int}(M)$ is contained in the interior of $\Pi^{-1}(E)$.
2. If ∂M is of class C^2 , then M is an immersion near its boundary and transverse to $\Pi^{-1}(C)$ along ∂M .

PROOF. The proof of this lemma uses the fact stated in Remark 2.2 that for every line $L \subset \mathbb{R}^2 \rtimes_A \{0\}$, the vertical plane $\Pi^{-1}(L)$ has zero mean curvature.

Suppose that M is a compact branched minimal surface with $\partial M \subset \Pi^{-1}(C)$ and we will prove the first item in the lemma. Arguing by contradiction, assume there exists a point $p \in \text{Int}(M)$ which is not contained in the interior of $\Pi^{-1}(E)$. Since E is convex, there exists a line $L \subset \mathbb{R}^2 \rtimes_A \{0\}$ such that $\Pi(p) \in L$ and L is disjoint from $\text{Int}(E)$. Hence the vertical minimal plane $\Pi^{-1}(L)$ intersects $\text{Int}(M)$ at p and so, by the maximum principle, M contains interior points on both sides of $\Pi^{-1}(L)$ near p .

Consider the product foliation $\mathcal{F}(L) = \{L_t\}_{t \in \mathbb{R}}$ of lines in $\mathbb{R}^2 \rtimes_A \{0\}$ parallel to $L = L_0$ and parameterized so that $E \subset \cup_{t \leq 0} L_t$. Let $\{\Pi^{-1}(L_t)\}_{t \in \mathbb{R}}$ be the related foliation of X by minimal vertical planes. By compactness of M , there is a largest value $t_0 > 0$, such that $\Pi^{-1}(L_{t_0}) \cap M \neq \emptyset$. But at any point of this non-empty intersection, we obtain a contradiction to the maximum principle applied to the minimal surfaces $\Pi^{-1}(L_{t_0})$ and M . This contradiction proves item 1 of the lemma. Item 2 of the lemma follows from Theorem 2 in [23]. \square

THEOREM 3.3. Let $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ be a metric semidirect product, E be a compact convex disk in $\mathbb{R}^2 \rtimes_A \{0\}$ and $C = \partial E$. Suppose $\Gamma \subset \Pi^{-1}(C)$ is a continuous simple closed curve such that the projection $\Pi: \Gamma \rightarrow C$ monotonically parameterizes C . Let $W = \Pi^{-1}(E)$. Then:

1. If D is a compact, branched minimal disk in X with $\partial D = \Gamma$, then the following properties hold:
 - 1a: D is an embedded disk.
 - 1b: The interior of D is a smooth Π -graph over the interior of E , i.e., $\Pi|_{\text{Int}(D)}: \text{Int}(D) \rightarrow \text{Int}(E)$ is a diffeomorphism.
 - 1c: If $\Pi|_{\Gamma}: \Gamma \rightarrow C$ is a homeomorphism, then $\Pi|_D: D \rightarrow E$ is a homeomorphism.
 - 1d: If Γ is of class C^2 , then the inclusion map of D is an immersion along ∂D and D is transverse to $\Pi^{-1}(C)$ along Γ .
 - 1e: If $\Pi|_{\Gamma}: \Gamma \rightarrow C$ is a diffeomorphism, then $\Pi|_D: D \rightarrow E$ is a diffeomorphism.
2. There exist compact minimal disks D_L, D_T, D_B in W with boundary Γ such that
 - 2a: D_L is an embedded disk of finite least area in X .
 - 2b: D_T is an embedded disk of finite least area in the closed region of W above the graph D_T .

¹This means that for every point $p \in C$, $\Pi^{-1}(p) \cap \Gamma$ is a compact interval or a single point.

- 2c: D_B is an embedded disk of finite least area in the closed region of W below the graph D_B .
 2d: Any compact branched minimal surface M in X whose boundary lies in the compact set $W(D_T, D_B) \subset W$ between the graphs D_T and D_B , satisfies $M \subset W(D_T, D_B)$. In particular, the disks D_T and D_B are uniquely defined by Properties 2b, 2c and 2d; hence, Γ is the boundary of a unique compact branched minimal surface if and only if $D_T = D_B$.

PROOF. We first prove item 1 of the theorem. Let D be a compact (possibly branched) minimal disk with boundary Γ . Consider D to be the image of a conformal harmonic map $f: \mathbb{D} \rightarrow X$, where \mathbb{D} is the closed unit disk in \mathbb{C} and $f|_{\partial\mathbb{D}}$ is a homeomorphism to Γ . To prove that $(\Pi \circ f)|_{\text{Int}(\mathbb{D})}: \text{Int}(\mathbb{D}) \rightarrow \text{Int}(E)$ is a diffeomorphism, we will modify a classical argument of Rado [30] who proved a similar result for minimal surfaces in \mathbb{R}^3 whose boundaries have an orthogonal injective projection to a convex planar curve. Since E is simply connected and $(\Pi \circ f)|_{\text{Int}(\mathbb{D})}: \text{Int}(\mathbb{D}) \rightarrow \text{Int}(E)$ is a proper map, to prove item 1b it suffices to check that the differential of $\Pi \circ f$ has rank two at every point of $\text{Int}(\mathbb{D})$. By contradiction, suppose that $p \in \text{Int}(\mathbb{D})$ is a point where the differential of $\Pi \circ f$ has rank less than two. In this case, either f is unbranched at p and the tangent plane $T_p D$ is vertical, or p is a branch point for f . We first consider the special case that f is unbranched at p and the tangent plane $T_p D$ is vertical, or equivalently, there exists a line $L \subset \mathbb{R}^2 \rtimes_A \{0\}$ passing through $(\Pi \circ f)(p) \in \text{Int}(E)$ such that the vertical plane $P = \Pi^{-1}(L)$ is tangent to D at the point $f(p)$.

The set $f^{-1}(P) \cap \text{Int}(\mathbb{D})$ is a 1-dimensional subset of \mathbb{D} that contains no isolated points (at regular points of f , this is a consequence of the maximum principle, while at branch points of f this follows from well known properties of branched minimal surfaces). The Π -projection of the boundary of $f[f^{-1}(P)]$ consists of two points in $L \cap C$ and $f^{-1}(P)$ has locally around $p \in \text{Int}(\mathbb{D})$ the appearance of a system of at least two analytic segments crossing at p (see e.g., Lemma 2 in Meeks and Yau [22]). Since $f(\text{Int}(\mathbb{D}))$ is a proper analytic (possibly branched) surface in $\text{Int}(W)$, then we conclude that $f^{-1}(P)$ contains the closure of the properly embedded analytic 1-complex $f^{-1}(P) \cap \text{Int}(\mathbb{D})$. Furthermore, $f^{-1}(P \cap \Gamma)$ consists of two components, each of which is a closed interval (possibly a point; this follows from the facts that $P \cap \Gamma$ consists of two components, $f(\text{Int}(\mathbb{D})) = \text{Int}(D) \subset \text{Int}(W)$ and $f|_{\partial\mathbb{D}}: \partial\mathbb{D} \rightarrow \Gamma$ is a homeomorphism), and the limit set of $f^{-1}(P) \cap \text{Int}(\mathbb{D})$ intersects both of these components. Note that each vertex in $f^{-1}(P) \cap \text{Int}(\mathbb{D})$ has a positive even number of associated edges with the number of edges at the vertex p being at least 4. As the component Δ of $f^{-1}(P) \cap \text{Int}(\mathbb{D})$ containing p has at least 4 ends or it contains a simple closed curve, then we conclude that either there is a simple closed curve α in Δ or there is a properly embedded arc α in Δ whose two ends are contained in the same component of $f^{-1}(P \cap \Gamma)$; in either case, there is a compact subset \mathbb{D}' of \mathbb{D} with non-empty interior bounded by α together with some connected subset of $f^{-1}(P \cap \Gamma) \subset \partial\mathbb{D}$. Consider the product foliation $\{L_t\}_{t \in \mathbb{R}}$ of lines in $\mathbb{R}^2 \rtimes_A \{0\}$ parallel to $L = L_0$. Let $\{\Pi^{-1}(L_t)\}_t$ be the related foliation of X by minimal vertical planes, and without loss of generality, suppose that $f(\mathbb{D}') \cap [\cup_{t>0} \Pi^{-1}(L_t)] \neq \emptyset$. Note that $f(\partial\mathbb{D}') \subset \Pi^{-1}(L)$. By compactness of \mathbb{D}' , there is a largest value $t_0 > 0$, such that $\Pi^{-1}(L_{t_0}) \cap f(\mathbb{D}') \neq \emptyset$. But at any point of this non-empty intersection, we obtain a contradiction with the maximum principle applied to the minimal surfaces $\Pi^{-1}(L_{t_0})$ and $f(\mathbb{D}')$. This contradiction proves that if p is not a branch point of f , then the differential of $\Pi \circ f$ has rank two at p .

On the other hand, if p is a branch point of f , then choose a horizontal line $L \subset \mathbb{R}^2 \rtimes_A \{0\}$ through $\Pi(p)$ so that the associated vertical plane $P = \Pi^{-1}(L)$ intersects Γ in exactly two points. Then, the set $f^{-1}(P) \cap \text{Int}(\mathbb{D})$ is a 1-dimensional subset of \mathbb{D} that contains no isolated points and $f^{-1}(P)$ has locally around $p \in \text{Int}(\mathbb{D})$ the appearance of a system of at least two analytic segments

crossing at p . Arguing as in the previous case gives a contradiction, which proves that f cannot have interior branch points; therefore, item 1b holds. Clearly item 1b implies 1a and 1c.

Note that if Γ is of class C^2 and $\Pi|_\Gamma: \Gamma \rightarrow C$ a C^2 -immersion, then by the statement and proof of Lemma 3.2, $\Pi|_D$ has rank two at every point of ∂D and D is an embedded disk transverse to ∂W along Γ . The remaining items of item 1 of the theorem follow directly from item 1b and this rank-two property of $\Pi|_D$ along ∂D .

We next prove item 2 of the theorem. By Theorem 1 in [23], there exists a disk D_L of finite least area in W with boundary Γ and every such least-area disk is an embedding. By Lemma 3.2, the interior of any compact branched minimal disk in X with $\partial D = \Gamma$ must be contained in the interior of W and so, any least-area disk in X with boundary Γ is contained in W . The existence of D_L proves item 2a of the theorem.

The existence of D_T can be found by constructing barriers. First suppose that Γ is smooth. Consider a compact branched minimal surface Σ in X with $\partial\Sigma = \Gamma$. By Lemma 3.2, $\Sigma \subset W$. Consider the closure C_Σ of the component of $W - \Sigma$ that contains a representative of the top end of W , by which we mean a closed region in W above some horizontal plane $\mathbb{R}^2 \times_A \{t_0\}$. Then Γ is homotopically trivial in C_Σ and by the barrier results in [23], Γ is the boundary of a finite least-area disk in C_Σ , which must be embedded (in fact, the interior of such a disk is a Π -graph over the interior of E by item 1b of this theorem); furthermore, any two such least-area disks in C_Σ intersect only along their common boundary Γ . Since this collection of 'disjoint' least-area embedded disks in C_Σ with boundary Γ forms a sequentially compact set (since they all have the same finite area in the homogeneously regular manifold X) and these disks are ordered by the relative heights of their graphing functions, then there exists a unique highest such least-area disk above Σ that we denote by $D_T(\Sigma)$. Approximation results in [22, 23] imply that when Γ is only continuous, then there also exists a similar embedded highest least-area disk $D_T(\Sigma)$ in C_Σ .

We claim that all the least-area disks $D_T(\Sigma)$ defined in the last paragraph lie in a compact region of W , independent of Σ . If $\text{trace}(A) = 0$ (equivalently, X is unimodular), then $\mathcal{F} = \{\mathbb{R}^2 \times_A \{z\} \mid z \in \mathbb{R}\}$ is a minimal foliation of X , and a simple application of the maximum principle to any compact branched minimal surface Σ' in X with boundary Γ and to the leaves of \mathcal{F} gives that $\max(z|_{\Sigma'}) \leq \max(z|_\Gamma)$, which proves the claim in this case. If X is non-unimodular, we can assume after scaling its metric that $\text{trace}(A) = 2$. Suppose that the claim fails to hold. Then, there exists a sequence of least-area disks $D_T(\Sigma_n) \subset W$ associated to compact branched minimal surfaces Σ_n , with $\partial\Sigma_n = \partial D_T(\Sigma_n) = \Gamma$ for all n and $\max(z|_{D_T(\Sigma_n)}) \rightarrow \infty$ as $n \rightarrow \infty$ (observe that the mean curvature comparison principle applied to $D_T(\Sigma_n)$ and to the leaves of \mathcal{F} ensures that $\min(z|_{D_T(\Sigma_n)}) \geq \min(z|_\Gamma)$). Given $n \in \mathbb{N}$, let $p_n \in D_T(\Sigma_n)$ be a point where $z|_{D_T(\Sigma_n)}$ attains its maximum value. By taking n large enough, we can assume that the intrinsic distance from p_n to Γ is greater than 2. As the $D_T(\Sigma_n)$ are stable, they have uniform curvature estimates away from their boundaries; in particular, the norm of the second fundamental form of $D_T(\Sigma_n)$ in the intrinsic ball of radius 1 centered at p_n is less than some positive constant C independent of n . This implies that there exists $\varepsilon > 0$ such that, for n large enough, the intrinsic disk $D(p_n, \varepsilon)$ of radius ε in $\mathbb{R}^2 \times_A \{z(p_n)\}$ centered at p_n satisfies the following property:

(P) Every vertical line passing through a point in $D(p_n, \varepsilon)$ intersects $D_T(\Sigma_n)$ near p_n .

Property (P) follows from the following observation, whose proof we leave to the reader:

(O) Let Σ_1, Σ_2 be two smooth surfaces in a homogeneous 3-manifold X that are tangent at a common point p , such that the intrinsic distance from p to the boundaries of these surfaces is at

least 1. If the norms of the second fundamental forms of these surfaces are less than some $C > 0$, then there exists an $\varepsilon = \varepsilon(C) \in (0, 1/2)$, less than the injectivity radius of X , such that every point in the intrinsic ball $B_{\Sigma_1}(p, \varepsilon) = \{q \in \Sigma_1 \mid d_{\Sigma_1}(p, q) < \varepsilon\}$ (here d_{Σ_1} denotes intrinsic distance in Σ_1) is a normal graph over a subdomain of the corresponding intrinsic ball $B_{\Sigma_2}(p, 2\varepsilon)$, of absolute value less than ε .

With property (P) at hand, we next find the desired contradiction that will prove our claim in the case X is non-unimodular. Since the area element dA_z for the restriction of the canonical metric to the plane $\mathbb{R}^2 \rtimes_A \{z\}$ is $dA_z = e^{-2z} dx \wedge dy$, then we conclude that

$$\text{Area}(E) \geq \text{Area}[\Pi(D(p_n, \varepsilon))] = e^{2z(p_n)} \text{Area}[D(p_n, \varepsilon)] = \pi \varepsilon^2 e^{2z(p_n)},$$

which implies that $z(p_n)$ is bounded from above, a contradiction. Now our claim is proved.

Once we know that all the least-area disks $D_T(\Sigma)$ lie in a compact region of W independent of Σ , then we conclude that they also have uniformly bounded area by the following argument: consider the disk $D_{z_0} := [\mathbb{R}^2 \rtimes_A \{z_0\}] \cap \Pi^{-1}(E)$ where $z_0 \gg 1$ is chosen sufficiently large so that $D_T(\Sigma)$ lies under D_{z_0} for every compact branched minimal surface Σ in X with boundary Γ . Consider the union D' of D_{z_0} with the annular portion of $\Pi^{-1}(\Gamma)$ below D_{z_0} and above Γ . D' clearly has finite area. Since $D_T(\Sigma)$ has least area among surfaces in the region of W above $D_T(\Sigma)$ with boundary Γ , then the area of D' is greater than or equal to the area of $D_T(\Sigma)$, as desired.

Given two of the disks, $D_T(\Sigma_1)$, $D_T(\Sigma_2)$, then using their union as a barrier, our previous arguments demonstrate that there is a least-area graphical disk D' with boundary Γ that lies in the region of $\Pi^{-1}(E)$ above both of them; here "above" means in the sense that the Π -graphing function $h: \text{Int}(E) \rightarrow \mathbb{R}$ for $\text{Int}(D')$ is greater than or equal to the graphing functions for the disks $D_T(\Sigma_1)$, $D_T(\Sigma_2)$. This notion of "above" induces a partial ordering on the set of disks of the form $D_T(\Sigma)$. A standard compactness argument using that the areas of these disks are uniformly bounded proves the existence of a minimal disk D_T with boundary Γ that is a maximal element in the partial ordering. By item 1a of this theorem, D_T is embedded, and by construction, D_T has least area among all compact surfaces in the closed region of W above D_T . This proves item 2b of the theorem. Item 2c about D_B can be proven by similar reasoning as in the proof of item 2b for D_T .

It remains to prove item 2d of the theorem. Suppose that M is a compact branched minimal surface in X whose boundary lies in the closed set $W(D_T, D_B) \subset W$ between the graphs D_T and D_B . Note that $M \subset W$ by the arguments in the proof of Lemma 3.2. Suppose that some point p of M lies in $W - W(D_T, D_B)$. First suppose that p lies in the closed region $D_T^+ \subset W$ that lies above D_T . Then using $D_T \cup (M \cap D_T^+)$ as a barrier, we obtain a minimal disk D'_T of least-area that lies above D_T , which contradicts that D_T is the highest disk that bounds Γ . This contradiction implies M does not intersect the interior of D_T^+ ; similar arguments imply that M does not intersect the interior of the region of W that lies below D_B . Hence, the main statement of item 2d is proved. Elementary separation arguments now imply that D_T and D_B are uniquely defined, and clearly if $D_T = D_B$, then every compact branched minimal surface in X with boundary Γ is equal to D_T . This completes the proof of Theorem 3.2. \square

By item 2d of Theorem 3.3, if a curve Γ satisfying the hypotheses of Theorem 3.3 is the boundary of a unique minimal disk, then it is also the boundary of a unique compact branched minimal surface. Our belief that graphical minimal disks bounding such a Γ are unique lead us to the following conjecture.

CONJECTURE 3.4. *Let $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ be a metric semidirect product. Suppose E is a compact convex disk in $\mathbb{R}^2 \rtimes_A \{0\}$, $C = \partial E$ and $\Gamma \subset \Pi^{-1}(C)$ is a continuous simple closed curve such that $\Pi: \Gamma \rightarrow C$ monotonically parameterizes C . Then the compact embedded disk D_L of finite least area given in Theorem 3.1 is the unique compact branched minimal surface in X with boundary Γ .*

The uniqueness property stated in Conjecture 3.4 is clear in the particular case that $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ is unimodular and Γ is contained in the plane $\mathbb{R}^2 \rtimes_A \{0\}$, by the maximum principle applied to $D_L = E$ and to the foliation of minimal planes $\{\mathbb{R}^2 \rtimes_A \{z\} \mid z \in \mathbb{R}\}$.

Next we prove the following particular case of Conjecture 3.4 for the case that X is non-unimodular.

PROPOSITION 3.5. *Let $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ be a non-unimodular metric semidirect product, and let $F_3(x, y, z) = F_3^H + \partial_z$ be the right invariant vector field in X given by (2.3). Suppose that $E \subset \mathbb{R}^2 \rtimes_A \{0\}$ is a compact convex disk with C^2 boundary Γ that is almost-transverse to F_3^H , in the sense that the inner product of the outward pointing unit conormal to E along Γ with F_3^H is greater than or equal to zero. Then, Γ is the boundary of a unique compact branched minimal surface which must therefore be the least-area, embedded minimal disk D_L given in Theorem 3.1.*

PROOF. Arguing by contradiction, suppose that the proposition fails to hold. Then item 2 of Theorem 3.1 implies that the embedded minimal disks D_T, D_B described there with boundary Γ are not equal. Let η_T, η_B denote the respective outward pointing unit conormals to these minimal disks along Γ . Since F_3 is a Killing vector field and D_T, D_B are minimal, then the divergence theorem gives

$$(3.1) \quad 0 = \int_{D_i} \operatorname{div}_{D_i}(F_3^{T_i}) = \int_{\Gamma} \langle \eta_i, F_3 \rangle = \int_{\Gamma} \langle \eta_i, F_3^H \rangle + \int_{\Gamma} \langle \eta_i, \partial_z \rangle,$$

where $i = T, B$ and $\operatorname{div}_{D_i}(F_3^{T_i})$ denotes the intrinsic divergence in D_i of the tangential component $F_3^{T_i}$ of F_3 to D_i .

On the other hand, observe that by the boundary maximum principle, D_B lies strictly below D_T near their common boundary Γ .

As X is non-unimodular, then $\mathbb{R}^2 \rtimes_A \{0\}$ has mean curvature 1, and so both D_T, D_B lie in $\mathbb{R}^2 \rtimes_A [0, \infty)$ (this follows from the interior maximum principle applied to D_i , $i = T, B$, and to $\mathbb{R}^2 \rtimes_A \{z_0\}$ for a suitable $z_0 < 0$) and D_T, D_B are transverse to $\mathbb{R}^2 \rtimes_A \{0\}$ along Γ (by the boundary maximum principle applied to D_i , $i = T, B$, and to $\mathbb{R}^2 \rtimes_A \{0\}$). Therefore, we deduce that

$$(3.2) \quad \langle \eta_T, \partial_z \rangle < \langle \eta_B, \partial_z \rangle < 0.$$

Expressing η_T as a sum of its horizontal and vertical components, we have

$$\eta_T = \langle \eta_T, \eta_H \rangle \eta_H + \langle \eta_T, E_3 \rangle E_3,$$

where η_H is the outward pointing unit conormal to E along Γ . As $\langle F_3^H, E_3 \rangle = 0$, then

$$\langle \eta_T, F_3^H \rangle = \langle \eta_T, \eta_H \rangle \langle \eta_H, F_3^H \rangle.$$

Note that $\langle \eta_H, F_3^H \rangle \geq 0$ since Γ is almost transverse to F_3^H , and that $\langle \eta_T, \eta_H \rangle \geq 0$ as D_T lies in $\Pi^{-1}(E)$. Thus $\langle \eta_T, F_3^H \rangle \geq 0$. Arguing similarly with D_B we will obtain $\langle \eta_B, F_3^H \rangle =$

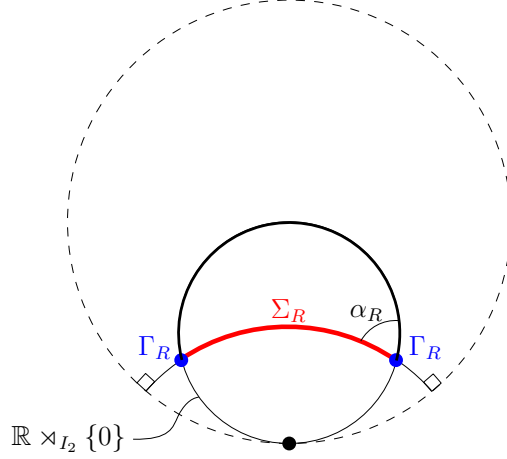


FIGURE 1. The constant angle α_R that the least-area disk Σ_R makes with the horosphere $\mathbb{R}^2 \rtimes_{I_2} \{0\}$ along $\partial\Sigma_R = \Gamma_R$ approaches $\pi/2$ as the Euclidean radius $R > 0$ of the circle Γ_R in the flat plane $\mathbb{R}^2 \rtimes_{I_2} \{0\}$ tends to infinity. Here we are using the Poincaré ball model of \mathbb{H}^3 , which is isometric to $\mathbb{R}^2 \rtimes_{I_2} \mathbb{R}$.

$\langle \eta_B, \eta_H \rangle \langle \eta_H, F_3^H \rangle$. As D_B lies below D_T near Γ , then $\langle \eta_T, \eta_H \rangle \leq \langle \eta_B, \eta_H \rangle$. Altogether we deduce that

$$(3.3) \quad 0 \leq \langle \eta_T, F_3^H \rangle \leq \langle \eta_B, F_3^H \rangle.$$

The inequalities (3.2) and (3.3) imply

$$\int_{\Gamma} \langle \eta_T, F_3^H \rangle + \int_{\Gamma} \langle \eta_T, \partial_z \rangle < \int_{\Gamma} \langle \eta_B, F_3^H \rangle + \int_{\Gamma} \langle \eta_B, \partial_z \rangle,$$

which contradicts (3.1). This contradiction proves the proposition. \square

4. Asymptotic behavior of certain compact minimal surfaces in non-unimodular metric Lie groups.

If I_2 is the identity matrix in $\mathcal{M}_2(\mathbb{R})$, then the metric Lie group $X = \mathbb{R}^2 \rtimes_{I_2} \mathbb{R}$ is isometric to hyperbolic 3-space, and the planes $\mathbb{R}^2 \rtimes_A \{t_0\}$ correspond to a family of horospheres with the same point at the ideal boundary of X . In this case, every circle Γ_R of Euclidean radius $R > 0$ in the (flat) plane $\mathbb{R}^2 \rtimes_{I_2} \{0\}$ is the boundary of a least-area compact disk $\Sigma_R \subset \mathbb{R}^2 \rtimes_{I_2} [0, \infty)$, which is contained in a totally geodesic hyperbolic plane H in X ; in fact, Σ_R is a geodesic disk of some hyperbolic radius in H . Furthermore, as the Euclidean radius of Γ_R in $\mathbb{R}^2 \rtimes_{I_2} \{0\}$ goes to infinity, then the Riemannian radius of Σ_R also goes to infinity and the constant angle that Σ_R makes with $\mathbb{R}^2 \rtimes_{I_2} \{0\}$ approaches $\pi/2$, see Figure 1. The next theorem shows that a similar result holds in any non-unimodular metric Lie group, since every such a non-unimodular metric Lie group is isomorphic and isometric to a non-unimodular semidirect product.

THEOREM 4.1. *Let $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ be a metric semidirect product, where $A \in \mathcal{M}_2(\mathbb{R})$ satisfies equation (2.9).*

1. *For each $\varepsilon > 0$, there exists a $\delta > 0$ such that the following property holds. Given a C^2 simple closed convex curve $\Gamma \subset \mathbb{R}^2 \rtimes_A \{0\}$ with geodesic curvature in $\mathbb{R}^2 \rtimes_A \{0\}$ less than δ in absolute value, then every compact branched minimal surface $M \subset X$ with $\partial M = \Gamma$ satisfies that*

- (a) $M \subset \mathbb{R}^2 \rtimes_A [0, \infty)$.
 - (b) $\text{Int}(M) \subset \Pi^{-1}(\text{Int}(E)) \cap [\mathbb{R}^2 \rtimes_A (0, \infty)]$, where E is the convex disk in $\mathbb{R}^2 \rtimes_A \{0\}$ bounded by Γ .
 - (c) $\langle \eta, \partial_z \rangle < -1 + \varepsilon$ along Γ , where η is the exterior unit conormal vector to M along its boundary.
2. Suppose that $\Gamma(n) \subset \mathbb{R}^2 \rtimes_A \{0\}$ is a sequence of C^2 simple closed convex curves with $\vec{0} = (0, 0, 0) \in \Gamma(n)$ such that the geodesic curvatures of $\Gamma(n)$ converge uniformly to 0 and the curves $\Gamma(n)$ converge on compact subsets to a line L with $\vec{0} \in L$ as $n \rightarrow \infty$. Then, for any sequence $M(n)$ of compact branched minimal disks (or compact stable minimal surfaces) with $\partial M(n) = \Gamma(n)$, the surfaces $M(n)$ converge C^2 on compact subsets of X to the vertical halfplane $\Pi^{-1}(L) \cap [\mathbb{R}^2 \rtimes_A [0, \infty)]$, as $n \rightarrow \infty$.

PROOF. We will first prove a key assertion which, together with properties of convex simple closed curves in \mathbb{R}^2 , will lead to the proof of the theorem.

ASSERTION 4.2. Given $\varepsilon \in (0, \frac{\pi}{2})$, there exists $R = R(\varepsilon) > 0$ such that for any $r > R$ the following property holds. Let $C_r \subset \mathbb{R}^2 \rtimes_A \{0\}$ be a circle of Euclidean radius $r > 0$ and let $D_B(r)$ be the minimal disk given by item 2c of Theorem 3.3 with boundary C_r . Then, the angle that the tangent plane to $D_B(r)$ makes with $\mathbb{R}^2 \rtimes_A \{0\}$ is greater than $\frac{\pi}{2} - \varepsilon$ at every point of C_r .

PROOF OF THE ASSERTION. Arguing by contradiction, suppose there exists $\varepsilon \in (0, \frac{\pi}{2})$, a sequence of circles $C_{r(n)} \subset \mathbb{R}^2 \rtimes_A \{0\}$ with Euclidean radii $r(n) \nearrow \infty$ and points $p_n \in C_{r(n)}$ such that the angle that the tangent plane to $D_B(r(n))$ at p_n makes with $\mathbb{R}^2 \rtimes_A \{0\}$ lies in $(0, \frac{\pi}{2} - \varepsilon]$. After left translating these disks we can assume that $p_n = \vec{0}$, and after choosing a subsequence, the $C_{r(n)}$ converge as $n \rightarrow \infty$ to a line $L \subset \mathbb{R}^2 \rtimes_A \{0\}$ and the closed disks $E_{r(n)} \subset \mathbb{R}^2 \rtimes_A \{0\}$ bounded by $C_{r(n)}$ converge to one of the two closed halfplanes bounded by L , which we denote by L^+ .

By item 2c of Theorem 3.3, given $n \in \mathbb{N}$ the unique compact embedded, stable minimal disk $D_B(r(n))$ associated to the circle $C_{r(n)}$ is a Π -graph over $E_{r(n)}$, and item 2d of the same theorem implies the following property:

(\star) Let W_n be the closure of the non-compact complement of $D_B(r(n)) \cup [\mathbb{R}^2 \rtimes_A \{0\}]$ in $\mathbb{R}^2 \rtimes_A [0, \infty)$. If $M \subset X$ is a compact, connected branched minimal surface whose boundary lies in W_n , then $M \subset W_n$.

To see why property (\star) above holds, we only need to observe the following: let $D_T(r(n))$ denote the compact embedded stable minimal disk with boundary $C_{r(n)}$ given in item 2b of Theorem 3.3. Note that $D_T(r(n))$ lies in W_n . Hence, if M is not contained in W_n , there must exist a compact piece M' of M such that $\partial M'$ lies in the region of $\mathbb{R}^2 \rtimes_A [0, \infty)$ bounded by $D_T(r(n))$ and $D_B(r(n))$, but M' has points outside that region. This would contradict item 2d of Theorem 3.3, which proves property (\star).

Since the circles $C_{r(n)}$ converge on compact subsets to the horizontal straight line L as $n \rightarrow \infty$, then a standard argument shows that after choosing a subsequence, the $D_B(r(n)) - C_{r(n)}$ converge to a minimal lamination \mathcal{L} of $X - L$ (local uniform curvature estimates for the $D_B(r(n))$ hold because these minimal surfaces are stable, see [32, 34]). \mathcal{L} is contained in $\Pi^{-1}(L^+) \cap [\mathbb{R}^2 \rtimes_A [0, \infty)]$, as $D_B(r(n)) - C_{r(n)} \subset \Pi^{-1}(E_{r(n)}) \cap [\mathbb{R}^2 \rtimes_A [0, \infty)]$ for every n and the $E_{r(n)}$ converge to L^+ as $n \rightarrow \infty$. It is clear that L is contained in the closure of \mathcal{L} . **We claim that $\overline{\mathcal{L}} \cap L^+ = L$:** if not, then there exists a point $p \in (\overline{\mathcal{L}} \cap L^+) - L$, such that p is the limit of a sequence of

points in $D_B(r(n))$. Consider a circle $C(p)$ of radius $\frac{1}{2}d(p, L)$ (d denotes Euclidean distance), and let $D_B(p)$ be the compact embedded, stable minimal disk with boundary $C(p)$ given by item 2c of Theorem 3.3. A straightforward adaptation of Property (\star) shows that for n sufficiently large, $D_B(r(n))$ lies in the closure of the non-compact complement of $D_B(p) \cup [\mathbb{R}^2 \rtimes_A \{0\}]$ in $\mathbb{R}^2 \rtimes_A [0, \infty)$. This contradicts that p is the limit of a sequence of points in $D_B(r(n))$, thereby proving our claim.

Since the $D_B(r(n))$ are all Π -graphs, then by standard arguments the leaves of \mathcal{L} are either Π -graphs over pairwise disjoint domains in $L^+ \subset \mathbb{R}^2 \rtimes_A \{0\}$, or they are contained in vertical half-planes. In particular, there exists a unique leaf of \mathcal{L} whose closure contains L . Since the disks $D_B(r(n))$ have uniform local ambient area bounds nearby their boundaries (this follows from the fact that $D_B(r(n))$ is least area in a certain region of $\mathbb{R}^2 \rtimes_A \mathbb{R}$, see the proof of Theorem 3.3), then after choosing a subsequence the surfaces with boundary $D_B(r(n))$ converge smoothly to a surface with boundary near L ; clearly, the interior of this limit surface is included in the unique leaf of \mathcal{L} that has L in its closure. Let D_∞ be the closure in $\mathbb{R}^2 \rtimes_A \mathbb{R}$ of such a leaf. The above arguments show that D_∞ is a smooth Π -graph with boundary L (smoothness of D_∞ holds up to its boundary). Observe that, by construction, the angle that the tangent plane to D_∞ at $\vec{0}$ makes with $\mathbb{R}^2 \rtimes_A \{0\}$ lies in $(0, \frac{\pi}{2} - \varepsilon]$.

Consider the right invariant vector field V on X such that $V(\vec{0})$ is unitary and tangent to L . **We claim that V is everywhere tangent to D_∞ .** Given $\tau \in L$, consider the angle $\theta(\tau) \geq 0$ that the Π -graph D_∞ makes with the plane $\mathbb{R}^2 \rtimes_A \{0\}$ at the point τ , viewed as a function $\theta: L \rightarrow [0, \pi/2]$. Recall that $\theta(\vec{0}) \in (0, \frac{\pi}{2} - \varepsilon]$ and observe that $\tau \in L \mapsto \theta(\tau)$ is analytic, as both X and D_∞ are analytic. If $\theta = \theta(\tau)$ is constant along L , then the unique continuation property of elliptic PDEs implies that for any $\tau \in L$, the left translation $\tau * D_\infty$ of D_∞ by τ is equal to D_∞ , which implies our claim. Hence for the remainder of the proof of our claim, we will assume that there is a $\tau \in L - \{\vec{0}\}$ such that $\theta(\vec{0}) \neq \theta(\tau)$.

(A) Suppose that $\theta(\vec{0}) > \theta(\tau)$. For $n \in \mathbb{N}$, consider a point σ_n of $\text{Int}(L^+)$ at Euclidean distance $\frac{1}{n}$ from τ . Then, the left translate $\sigma_n * D_B(r(n))$ of $D_B(r(n))$ by σ_n has boundary $\sigma_n * C_{r(n)}$. For $n \in \mathbb{N}$ fixed, there exists an integer $j(n) > n$ such that for all $j \in \mathbb{N}$ with $j \geq j(n)$, the boundary $\partial(\sigma_n * D_B(r(n))) = \sigma_n * C_{r(n)}$ lies in the interior of the disk $E_{r(j)}$. By Property (\star) applied to $\sigma_n * D_B(r(n))$ and to $M = D_B(r(j))$, we deduce that $D_B(r(j))$ lies in the closure W_n of the non-compact complement of $[\sigma_n * D_B(r(n))] \cup [\mathbb{R}^2 \rtimes_A \{0\}]$ in $\mathbb{R}^2 \rtimes_A [0, \infty)$. Equivalently, the graphing functions $v_n: \sigma_n * E_{r(n)} \rightarrow \mathbb{R}$, $u_j: E_{r(j)} \rightarrow \mathbb{R}$ such that $\sigma_n * D_B(r(n))$ (resp. $D_B(r(j))$) is the Π -graph of v_n (resp. of u_j), satisfy $v_n \leq u_j$ in $\sigma_n * E_{r(n)}$, for every $j \geq j(n)$. After taking limits as $j \rightarrow \infty$ (but letting n fixed), we conclude that $\sigma_n * D_B(r(n))$ lies below D_∞ . Taking limits as $n \rightarrow \infty$, we have that $\tau * D_\infty$ lies below D_∞ . In particular, the angle that $\tau * D_\infty$ makes with the plane $\mathbb{R}^2 \rtimes_A \{0\}$ at τ (which equals $\theta(\vec{0})$) cannot be greater than the angle that D_∞ makes with $\mathbb{R}^2 \rtimes_A \{0\}$ at τ (which is $\theta(\tau)$), a contradiction with our hypothesis in this case (A).

(B) We next perform slight modifications in the arguments in (A) to find a contradiction in the case that $\theta(\vec{0}) < \theta(\tau)$. Given $n \in \mathbb{N}$ large, let $\sigma_n \in \text{Int}(E_{r(n)})$ be the point at distance $1/n$ from $\vec{0}$ so that the segment $[\vec{0}, \sigma_n]$ with end points $\vec{0}$ and σ_n is orthogonal to $C_{r(n)}$ at $\vec{0}$. Arguing as in case (A), there exists an integer $j(n) > n$ such that for every $j \in \mathbb{N}$, $j \geq j(n)$, the left translate of $C_{r(n)}$ by σ_n lies in the interior of $E_{r(j)}$. By Property (\star) applied to $\sigma_n * D_B(r(n))$ and to $M = \tau * D_B(r(j))$, we deduce that $\tau * D_B(r(j))$ lies in the closure W_n of the non-compact

complement of $[\sigma_n * D_B(r(n))] \cup [\mathbb{R}^2 \rtimes_A \{0\}]$ in $\mathbb{R}^2 \rtimes_A [0, \infty)$. Equivalently, the graphing functions $v_n: \sigma_n * E_{r(n)} \rightarrow \mathbb{R}$, $u_j: \tau * E_{r(j)} \rightarrow \mathbb{R}$ such that $\sigma_n * D_B(r(n))$ (resp. $\tau * D_B(r(j))$) is the Π -graph of v_n (resp. of u_j), satisfy $v_n \leq u_j$ in $\sigma_n * E_{r(n)}$, for every $j \geq j(n)$. Taking limits as $j \rightarrow \infty$ with n fixed, we conclude that $\sigma_n * D_B(r(n))$ lies below $\tau * D_\infty$. Taking limits as $n \rightarrow \infty$, we have that D_∞ lies below $\tau * D_\infty$. In particular, the angle that D_∞ makes with the plane $\mathbb{R}^2 \rtimes_A \{0\}$ at τ (which equals $\theta(\tau)$) cannot be greater than the angle that $\tau * D_\infty$ makes with $\mathbb{R}^2 \rtimes_A \{0\}$ at τ (which is $\theta(\vec{0})$), that is, $\theta(\tau) \leq \theta(\vec{0})$, which is contrary to our hypothesis.

From (A) and (B) we conclude the proof of our claim that V is everywhere tangent to D_∞ .

Recall that V is horizontal and right invariant. It follows from equation (2.3) that in (x, y, z) -coordinates in $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$, V is a linear combination of ∂_x, ∂_y with constant coefficients, and thus, its integral curves are horizontal lines, all parallel to L in the Euclidean sense. Since V is everywhere tangent to D_∞ , then the integral curves of V passing through points in D_∞ are completely contained in D_∞ . Therefore, D_∞ is foliated by these horizontal lines. Let $L^\perp \subset \mathbb{R}^2 \rtimes_A \{0\}$ be the straight line orthogonal to L passing through the origin. Since D_∞ is a minimal Π -graph and it is foliated by straight lines parallel to L , then the intersection of D_∞ with the vertical plane $\Pi^{-1}(L^\perp)$ is a proper analytic arc γ with $\vec{0} \in \gamma$, and γ is a Π -graph over its projection to L^\perp . Note that the z -coordinate restricted to γ cannot have a local minimum value z_0 , by the mean curvature comparison principle applied to the minimal surface D_∞ and to the mean curvature one surface $\mathbb{R}^2 \rtimes_A \{z_0\}$. Since γ is analytic, if γ is not parameterized by its z -coordinate, then $z|_\gamma$ must have a first local maximum. As γ lies above $\mathbb{R}^2 \rtimes_A \{0\}$, then γ must be asymptotic to a horizontal line at height $z_1 \geq 0$ and thus, D_∞ is smoothly asymptotic to $\mathbb{R}^2 \rtimes_A \{z_1\}$. This contradicts that D_∞ is minimal and $\mathbb{R}^2 \rtimes_A \{z_1\}$ has mean curvature one. This contradiction shows that γ can be parameterized by its z -coordinate; in fact, the range of values of z along γ is $[0, \infty)$ since D_∞ cannot be smoothly asymptotic to any horizontal plane $\mathbb{R}^2 \rtimes_A \{z_2\}$ for any $z_2 > 0$. In what follows we will parameterize γ by the height $z \in [0, \infty)$.

To obtain the contradiction that will prove Assertion 4.2, we apply a flux argument to an appropriate annular quotient of D_∞ . For any $t \in (0, \infty)$, consider the minimal strip

$$D_\infty(t) = D_\infty \cap (\mathbb{R}^2 \rtimes_A [0, t]).$$

Fix $q \in L - \{\vec{0}\}$ and consider the infinite cyclic subgroup \mathcal{I} of isometries of X generated by the left translation by q . Then X/\mathcal{I} is a homogeneous 3-manifold diffeomorphic to $\mathbb{S}^1 \times \mathbb{R}^2$, the z -coordinate on X descends to a well-defined function on X/\mathcal{I} (which we will also call the height z), and every right invariant horizontal vector field F on X descends to a well-defined Killing field \widehat{F} on X/\mathcal{I} . Consider the quotient minimal annulus $\Omega(t) = D_\infty(t)/\mathcal{I}$ in X/\mathcal{I} .

First consider the case that the Milnor D -invariant of X is positive. We next prove that the length in X/\mathcal{I} of the boundary curve c_t of $\Omega(t)$ at height t converges to zero exponentially quickly as $t \rightarrow \infty$. To see this, without loss of generality we may assume that L points in the x -direction (after a rotation in the (x, y) -plane, which does not change the ambient left invariant metric but does change the matrix A). Then, $q = (x(q), 0, 0)$ with $x(q) \in \mathbb{R} - \{0\}$ and equation (2.7) gives that

$$\text{length}(c_t) = \int_0^{x(q)} \|\partial_x\| dx = \int_0^{x(q)} \sqrt{a_{11}(-t)^2 + a_{21}(-t)^2} dx = \|\partial_x\|(0, 0, t) |x(q)|,$$

where $(a_{ij}(z))_{i,j}$ denotes the matrix $e^{zA} \in \mathcal{M}_2(\mathbb{R})$. Now we deduce that $\text{length}(c_t)$ converges to zero exponentially quickly as $t \rightarrow \infty$ from item 2a of Proposition 6.1 in the Appendix, as the

Milnor D -invariant is positive. The same item 2a of Proposition 6.1 ensures that all horizontal right invariant vector fields in X have lengths decaying exponentially as $z \rightarrow +\infty$. Pick a right invariant horizontal vector field F in X such that F, V are linearly independent. Let \hat{F} be the quotient Killing field of F in X/\mathcal{I} . Consider for each $t \geq 0$ the flux of \hat{F} across c_t , defined as

$$\text{Flux}(\hat{F}, c_t) = \int_0^{x(q)} \langle \hat{F}, \eta \rangle dx,$$

where η is the unit vector field which is tangent to D_∞/\mathcal{I} , orthogonal to c_t with $\langle \eta, E_3 \rangle \geq 0$. Since \hat{F} has constant length along c_t with this constant being bounded as a function of $t > 0$, and the length of c_t converge to zero as $t \rightarrow \infty$, then $\text{Flux}(\hat{F}, c_t)$ also converges to zero as $t \rightarrow \infty$. As $\text{Flux}(\hat{F}, c_0)$ is non-zero and $\text{Flux}(\hat{F}, c_t)$ is independent of t (because the divergence of the tangential component of \hat{F} along $\Omega(t)$ is zero), then we obtain a contradiction. This contradiction completes the proof of Assertion 4.2 in the case $D > 0$.

If $D \leq 0$, then item 2b of Proposition 6.1 gives that A is diagonalizable with one positive eigenvalue λ and another non-positive eigenvalue μ . In this case, there exists a horizontal right invariant vector field F on X with $\|F(\vec{0})\| = 1$, such that the norm of F decreases exponentially quickly as $z \rightarrow +\infty$ (namely, F is determined by $F(\vec{0})$ being the unitary eigenvector of the matrix A associated to λ , since $\|F\|(z) = e^{-\lambda z}$ by item 2b of Proposition 6.1). Let \hat{F} be the quotient Killing field of F on X/\mathcal{I} . Suppose for the moment that F is not collinear with V . As before, we may assume that L points in the x -direction and $q = (x(q), 0, 0)$. Then, the flux of \hat{F} along c_t satisfies

$$|\text{Flux}(\hat{F}, c_t)| \leq \int_0^{x(q)} \|F\|(x_0, y_0, t) dx.$$

But the line element dx grows at most exponentially as $z \rightarrow +\infty$, and in fact at most as the function $e^{-\mu z}$. Since $2 = \text{trace}(A) = \lambda + \mu$, then $\|F\| dx$ decays exponentially as $z \rightarrow +\infty$ and thus, we arrive to a contradiction as in the former case $D > 0$.

The last case we must consider is $D \leq 0$ and F is proportional to V . In this case, we still normalize L to point in the direction of the x -axis, and replace F in the above computations by ∂_y . Then dx decays like $e^{-\lambda z}$ (with the same notation as before) while $\|F\|$ increases at most like $e^{-\mu z}$ as $z \rightarrow +\infty$, and the conclusion is the same. Now Assertion 4.2 is proved. \square

We now prove item 1 in the statement of Theorem 4.1. Let Γ be a simple closed convex curve contained in $\mathbb{R}^2 \rtimes_A \{0\}$, and let $M \subset X$ be a compact branched minimal surface with $\partial M = \Gamma$. By Lemma 3.2, $\text{Int}(M) \subset \Pi^{-1}(\text{Int}(E))$, where E is the convex disk in $\mathbb{R}^2 \rtimes_A \{0\}$ bounded by Γ . If $\text{Int}(M)$ is not contained in $\mathbb{R}^2 \rtimes_A (0, \infty)$, then there exists a point $p \in \text{Int}(M)$ with smallest non-positive z -coordinate z_0 . An application of the mean curvature comparison principle gives a contradiction to the fact that the mean curvature of the plane $\mathbb{R}^2 \rtimes_A \{z_0\}$ is 1 and M lies on the mean convex side of this plane at the point p . Therefore, conditions (a), (b) in item 1 of Theorem 4.1 hold.

Next we prove that condition (c) in item 1 holds. Arguing by contradiction, suppose that there exists an $\varepsilon > 0$, a sequence $M(n)$ of compact branched minimal surfaces with C^2 simple closed convex boundary curves $\Gamma(n) \subset \mathbb{R}^2 \rtimes_A \{0\}$, and points $p_n \in \Gamma(n)$ such that

(C1) The geodesic curvature of $\Gamma(n)$ is uniformly going to zero as $n \rightarrow \infty$.

(C2) $\langle \eta_n(p_n), \partial_z(p_n) \rangle \geq -1 + \varepsilon$ for all $n \in \mathbb{N}$, where η_n is the exterior unit conormal vector to $M(n)$ along $\Gamma(n)$ (observe that in order to make sense of η_n we are using that $M(n)$ is an immersion near $\Gamma(n)$ by item 2 of Lemma 3.2).

Let $R = R(\varepsilon)$ be the positive number produced by Assertion 4.2 applied to the value $\varepsilon > 0$ that appears in Condition (C2) above. By Condition (C1), for n large enough we can assume that the geodesic curvature of $\Gamma(n)$ is less than $1/R$. This property allows us to find a round disk $\widehat{E}_n \subset \mathbb{R}^2 \rtimes_A \{0\}$ of radius R that is contained in the disk $E(n)$ bounded by $\Gamma(n)$ and such that $\widehat{E}_n \cap E(n) = \{p_n\}$. Let $\widehat{\Gamma}(n) = \partial \widehat{E}_n$ and let $D_B(n)$ be the ‘lowest’ minimal disk bounded by $\widehat{\Gamma}(n)$ given by item 2c of Theorem 3.3. Since $D_B(n)$ lies below the branched minimal surface $M(n)$ (this follows from Property (\star) in the proof of Assertion 4.2 applied to $D_B(n)$ and $M(n)$) and $D_B(n) \cap M(n) = \{p_n\}$, then the angle that the tangent plane $T_{p_n} M(n)$ makes with $\mathbb{R}^2 \rtimes_A \{0\}$ is greater than the angle φ_n that $T_{p_n} D_B(n)$ makes with $\mathbb{R}^2 \rtimes_A \{0\}$. Since Condition (C1) and Assertion 4.2 allow us to take φ_n arbitrarily close to $\pi/2$ for n sufficiently large, then we conclude that the angle that $T_{p_n} M(n)$ makes with $\mathbb{R}^2 \rtimes_A \{0\}$ converges to $\pi/2$ as $n \rightarrow \infty$. This contradiction completes the proof of item 1c of the theorem.

We next prove item 2 of the theorem. Suppose that $\Gamma(n) \subset \mathbb{R}^2 \rtimes_A \{0\}$ is a sequence of C^2 simple closed convex curves with $\vec{0} \in \Gamma(n)$, having geodesic curvatures uniformly approaching 0 as $n \rightarrow \infty$ and converging on compact subsets to a straight line L that contains $\vec{0}$. Let $M(n)$ be a sequence of compact branched minimal disks (or compact stable minimal surfaces) with $\partial M(n) = \Gamma(n)$. Suppose for the moment that the $M(n)$ are disks; we will discuss later the changes necessary to prove the case that the $M(n)$ are stable. By item 1 of Theorem 3.3, the disks $M(n)$ are unbranched and Π -graphs over the compact convex disks $E(n)$ bounded by $\Gamma(n)$ in $\mathbb{R}^2 \rtimes_A \{0\}$. We claim that the $M(n)$ have uniformly bounded second fundamental forms up to their boundaries; to see this, suppose this property fails. Left translate the $M(n)$ so that the norm of the second fundamental form is largest at the origin, and rescale the (x, y, z) -coordinates by this maximum norm of the second fundamental form of the $M(n)$, obtaining a new sequence of rescaled minimal Π -graphs with uniformly bounded second fundamental form. After extracting a subsequence, these rescaled Π -graphs converge to a non-flat minimal surface M_∞ in \mathbb{R}^3 possibly with boundary (if ∂M_∞ is non-empty then ∂M_∞ is a horizontal straight line and M_∞ lies entirely above the horizontal plane that contains ∂M_∞). Note that the Gaussian image of M_∞ is contained in the closed upper hemisphere, which is clearly impossible if some component of M_∞ has empty boundary (note that this component would be complete). This implies that M_∞ is connected and has nonempty boundary. It follows that M_∞ is a graphical stable minimal surface in the closed upper half-space of \mathbb{R}^3 bounded by the horizontal plane that contains ∂M_∞ , which can also be easily ruled out, since M_∞ together with its image under the 180° -rotation around ∂M_∞ is a complete, non-flat minimal graph. Therefore, the $M(n)$ have uniformly bounded second fundamental forms up to their boundaries.

It follows that a subsequence of the $M(n)$ (denoted in the same way) converges as $n \rightarrow \infty$ on compact subsets of X to a minimal lamination \mathcal{L} of $X - L$, and \mathcal{L} contains a leaf M_∞ with boundary the straight line L . Since the geodesic curvatures of the curves $\Gamma(n)$ converge uniformly to 0 as $n \rightarrow \infty$, then item 1 of this theorem implies that $\langle \eta_n, \partial_z \rangle$ is arbitrarily close to -1 for n large enough (here η_n is the exterior conormal vector field to $M(n)$ along $\Gamma(n)$). It follows that the limit surface M_∞ is tangent to the closed vertical halfplane $\Pi^{-1}(L) \cap [\mathbb{R}^2 \rtimes_A [0, \infty)]$ along L . Since the $M(n)$ all lie at one side of $\Pi^{-1}(L) \cap [\mathbb{R}^2 \rtimes_A [0, \infty)]$, then M_∞ also lies at one side of $\Pi^{-1}(L) \cap [\mathbb{R}^2 \rtimes_A [0, \infty)]$ and thus, the boundary maximum principle implies that

$M_\infty = \Pi^{-1}(L) \cap [\mathbb{R}^2 \rtimes_A [0, \infty)]$. We now prove that \mathcal{L} contains no other leaves different from M_∞ . Arguing by contradiction, any other leaf component Σ of \mathcal{L} must be a complete positive Π -graph (without boundary) over its projection to $\mathbb{R}^2 \rtimes_A \{0\}$, and Σ has bounded second fundamental form by arguments in the previous paragraph. But the existence of such a graphical leaf Σ in $\mathbb{R}^2 \rtimes_A (0, \infty)$ is easily seen to be impossible by considering its behavior on a sequence of points $p_k = (x_k, y_k, z_k) \in \Sigma$ where $\lim_{k \rightarrow \infty} z_k$ is the infimum $z_0 \geq 0$ of the z -coordinate function of Σ (recall that the minimal surface Σ cannot be asymptotic to the mean curvature one surface $\mathbb{R}^2 \rtimes_A \{z_0\}$). This contradiction proves that $\mathcal{L} = \{M_\infty\}$, and thus, a subsequence of the $M(n)$ converges to the desired halfplane. Since every subsequence of the $M(n)$ has a convergent subsequence which equals this limit, then the entire sequence $M(n)$ converges to $\Pi^{-1}(L) \cap [\mathbb{R}^2 \rtimes_A [0, \infty)]$. This completes the proof of item 2 of the theorem in the case that the $M(n)$ are disks.

If the $M(n)$ are compact stable surfaces (not disks) then the curvature estimates by Schoen [34] and Ros [32] give that the $M(n)$ have uniformly bounded second fundamental forms away from their boundaries. As previously, for each $n \in \mathbb{N}$ let $E(n)$ be the convex compact disk bounded by $\Gamma(n)$ in $\mathbb{R}^2 \rtimes_A \{0\}$. Note that by barrier arguments as in the proof of items 2b, 2c in Theorem 3.3, for each $n \in \mathbb{N}$ there exists a least-area disk $D(n)$ with boundary $\Gamma(n)$ in the closure of the bounded region of $[\mathbb{R}^2 \rtimes_A [0, \infty)] - M(n)$ that contains $E(n)$. Furthermore, $D(n) \subset \Pi^{-1}(E(n)) \cap [\mathbb{R}^2 \rtimes_A [0, \infty)]$ is a Π -graph over $E(n)$. Also, $M(n)$ lies “above” the Π -graph $D(n)$. As the previously considered case of disks ensures that the $D(n)$ converge to $\Pi^{-1}(L) \cap [\mathbb{R}^2 \rtimes_A [0, \infty)]$ as $n \rightarrow \infty$, then the $M(n)$ converge (as sets) to $\Pi^{-1}(L) \cap [\mathbb{R}^2 \rtimes_A [0, \infty)]$. We now check that the last convergence is of class C^2 , by showing that the $M(n)$ have uniformly bounded second fundamental form up to their boundaries (this would finish the proof of item 2 of Theorem 4.1 in this case). If this is not the case, then the rescaling-by-curvature argument above produces a limit of a subsequence of the $M(n)$ which is a non-flat, stable minimal surface M_∞ in \mathbb{R}^3 , such that either has no boundary, or M_∞ has non-empty boundary given by a horizontal line and M_∞ is contained in a quarter of space $Q \subset \mathbb{R}^3$ with ∂M_∞ being the set of non-smooth points of ∂Q . If $\partial M_\infty = \emptyset$, then M_∞ is complete, which contradicts that M_∞ is non-flat and stable. Therefore, $\partial M_\infty \neq \emptyset$. In this case, the rescaled least-area disks $D(n)$ are all below the related rescaled $M(n)$. Since these rescaled images of $D(n)$ converge as $n \rightarrow \infty$ to a vertical half-plane in Q , then M_∞ must be equal to this vertical limit half-plane, which contradict the non-flatness of M_∞ . Now the theorem is proved. \square

COROLLARY 4.3. *Let $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ be a metric semidirect product, where $A \in \mathcal{M}_2(\mathbb{R})$ satisfies equation (2.9). Then, there exists a straight line $L \subset \mathbb{R}^2 \rtimes_A \{0\}$ with $\vec{0} \in L$ such that the following property holds.*

(Q) *Let $p, q \in L$ be different points with $\vec{0} \in (p, q)$ (here $(p, q) \subset L$ is the open segment with extrema p, q), and let $C_p, C_q \subset \mathbb{R}^2 \rtimes_A \{0\}$ be pairwise disjoint Euclidean circles centered at points in $L - [p, q]$, with $p \in C_p, q \in C_q$. If the Euclidean radii of C_p, C_q are sufficiently large, then there exists an embedded least-area annulus $\Sigma \subset \mathbb{R}^2 \rtimes_A [0, \infty)$ with boundary $\partial \Sigma = C_p \cup C_q$ (see Figure 2).*

Furthermore:

- (1) *If the Milnor D -invariant of X is $D > 0$, then property (Q) holds for every line $L \subset \mathbb{R}^2 \rtimes_A \{0\}$ with $\vec{0} \in L$.*
- (2) *If $D \leq 0$, then property (Q) holds for the line $L \subset \mathbb{R}^2 \rtimes_A \{0\}$ with $\vec{0} \in L$ in the direction of the eigenvector of A associated to a positive eigenvalue.*

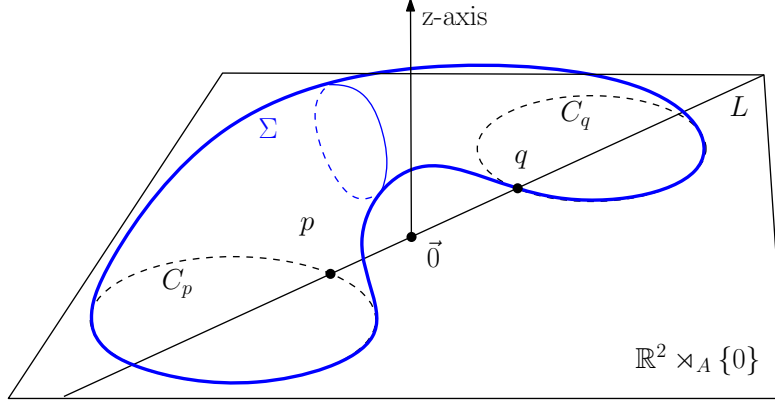


FIGURE 2. The minimal annulus Σ that appears in Corollary 4.3, for a pair of pairwise disjoint, large enough circles $C_p, C_q \subset \mathbb{R}^2 \rtimes_A \{0\}$.

PROOF. Suppose first that $D > 0$, and let $L \subset \mathbb{R}^2 \rtimes_A \{0\}$ be any line with $\vec{0} \in L$. By item 2a in Proposition 6.1, the vertical lines $l_p = \{(p, z) \mid z \in \mathbb{R}\}$ and $l_q = \{(q, z) \mid z \in \mathbb{R}\}$ are both asymptotic to the z -axis as $z \rightarrow \infty$. Now consider pairwise disjoint Euclidean circles $C_p, C_q \subset \mathbb{R}^2 \rtimes_A \{0\}$ centered at points in $L - [p, q]$, with $p \in C_p, q \in C_q$. Let D_p, D_q be compact, embedded, least-area disks with boundaries $\partial D_p = C_p, \partial D_q = C_q$, which exist by Theorem 3.3. By Assertion 4.2, if the Euclidean radii of C_p, C_q is sufficiently large, then D_p, D_q are arbitrarily close to the vertical half-planes $\Pi^{-1}(L_p) \cap [\mathbb{R}^2 \rtimes_A [0, \infty)]$, $\Pi^{-1}(L_q) \cap [\mathbb{R}^2 \rtimes_A [0, \infty)]$, where $L_p, L_q \subset \mathbb{R}^2 \rtimes_A \{0\}$ are the lines orthogonal to L that pass through p, q respectively. Since $l_p \subset L_p$ and $l_q \subset L_q$ are asymptotic to the z -axis as $z \rightarrow \infty$, then l_p and l_q are asymptotic to each other, and thus the distance between the disjoint area minimizing disks D_p and D_q is arbitrarily small if the Euclidean radii of C_p, C_q are sufficiently large. Therefore, after replacing a pair of intrinsic geodesic disks $D'_p \subset D_p, D'_q \subset D_q$ of radius 1 centered at sufficiently close points of D_p, D_q by an annulus of least area with boundary $\partial D'_p \cup \partial D'_q$, we obtain a piecewise smooth annulus with area less than the sum of the areas of the least-area disks D'_p, D'_q . By the Douglas criterion (the area of some annulus bounding $C_p \cup C_q$ is less than the infimum of the areas of any two disks bounding $C_p \cup C_q$), there exists by Morrey [25] an annulus Σ of least area in X with boundary $C_p \cup C_q$. Note that $\text{Int}(\Sigma) \subset \mathbb{R}^2 \rtimes [0, \infty)$ by the maximum principle applied to Σ and to planes $\mathbb{R}^2 \rtimes_A \{z\}$ with $z < 0$, then by the Geometric Dehn's Lemma for Planar Domains given in Theorem 5 in [22], Σ is a smooth embedded annulus (actually, Theorem 5 in [22] is stated for three-manifolds with convex boundary but the convex boundary is only used to obtain the existence of a least-area immersed annulus, which we already have in this case).

Suppose $D \leq 0$. As the eigenvalues of A are the roots of the polynomial $\lambda^2 - 2\lambda + D = 0$, then $\lambda = 1 \pm \sqrt{1 - D}$. Hence, exactly one of these eigenvalues λ_+ is greater than or equal to 2, and the other one is non-positive. After an orthogonal change of basis (that does not change the metric Lie group structure of X), the matrix A transforms to $A_1 = OAO^{-1}$ for some orthogonal matrix O , where A_1 has entries $a_{11} = \lambda_+$ and $a_{21} = 0$, and having an associated eigenvector $(1, 0)$. Consider the line $L = \{(t, 0, 0)\} \subset \mathbb{R}^2 \rtimes_{A_1} \{0\}$. In this case, equation (2.4) shows that $E_1 = e^{\lambda_+ z} \partial_x$, and so, $\|\partial_x\|$ is exponentially decaying as $z \rightarrow \infty$. Hence, for any pair of different points $p, q \in L$, the vertical lines $l_p = \{(p, z) \mid z \in \mathbb{R}\}$ and $l_q = \{(q, z) \mid z \in \mathbb{R}\}$ are both asymptotic to the z -axis as $z \rightarrow \infty$. Now consider pairwise disjoint Euclidean circles $C_p, C_q \subset \mathbb{R}^2 \rtimes_{A_1} \{0\}$ centered at

points in $L - [p, q]$, with $p \in C_p$, $q \in C_q$. Let D_p, D_q be compact, embedded, least-area disks with boundaries $\partial D_p = C_p$, $\partial D_q = C_q$, which exist by Theorem 3.3. Arguing as in the previous paragraph, if the Euclidean radii of C_p, C_q are sufficiently large, there exists an embedded least-area annulus $\Sigma \subset \mathbb{R}^2 \rtimes_A [0, \infty)$ with boundary $\partial \Sigma = C_p \cup C_q$, and the proof is complete. \square

5. Radius estimates for cylindrically bounded stable minimal surfaces.

In this section we obtain radius estimates for compact stable minimal surfaces in semidirect products, using the results from Section 4. Given $r > 0$ and a vertical geodesic Γ in a metric semidirect product $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$, we will denote by $\mathcal{W}(\Gamma, r) \subset X$ the closed solid metric cylinder of radius r centered along Γ .

PROPOSITION 5.1. *Let $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ be a metric semidirect product, where A is either as in equation (2.9) with Milnor D -invariant less than 1 or where $\text{trace}(A) = 0$ (this is the case where X is unimodular). For every vertical geodesic $\Gamma \subset X$ and $r > 0$, there exists a $j \in \mathbb{N}$ such that every compact immersed minimal surface M in X with $\partial M \subset \mathcal{W}(\Gamma, r)$ satisfies $M \subset \mathcal{W}(\Gamma, jr)$.*

PROOF. Without loss of generality, we will henceforth assume that Γ is the z -axis. Recall that if X is unimodular, then it is isomorphic to $\mathbb{R}^3, \tilde{E}(2), \text{Nil}_3$ or Sol_3 .

First suppose that X is not isomorphic to $\tilde{E}(2)$ or Nil_3 . By Theorem 3.6 in [21] (see also Examples 3.2–3.5 therein), there are two distinct vertical planes P_1, P_2 (in (x, y, z) -coordinates, in fact, P_1 can be taken as the (x, z) -plane and P_2 as the (y, z) -plane) that are Lie subgroups of X . For $i = 1, 2$, let $U_i(R)$ be the closed regular neighborhood of P_i of radius $R > 0$. **We claim that the boundary surfaces $\partial U_i(R) = P_i^+(R) \cup P_i^-(R)$ of $U_i(R)$ both have non-negative mean curvature in X with respect to the inward pointing normal to $U_i(R)$.** Since each of $P_i^+(R), P_i^-(R)$ are at constant distance from P_i , which is a connected, codimension-one subgroup in X , then Lemma 3.9 in [21] implies that $P_i^\pm(R)$ is a right coset of P_i and is also a left coset of some 2-dimensional subgroup $\Sigma_i^\pm(R)$ of X . This last property implies that $P_i^\pm(R)$ has constant mean curvature, as every 2-dimensional subgroup in a metric Lie group has this property. Hence it remains to show that the mean curvature vector of $P_i^\pm(R)$ points towards $U_i(R)$. Note that $\Sigma_i^\pm(R)$ must be disjoint from $P_i^\pm(R)$ (otherwise $\Sigma_i^\pm(R) = P_i^\pm(R)$, which implies $\vec{0} \in P_i^\pm(R) \cap P_i$ hence $R = 0$, a contradiction). In this situation, the classification of codimension-one subgroups in Theorem 3.6 in [21] implies that $\Sigma_i^\pm(R)$ is one of the elements in the 1-parameter family \mathcal{A}_i of 2-dimensional subgroups of X that share the 1-dimensional subgroup $P_i \cap [\mathbb{R}^2 \rtimes_A \{0\}]$ (also called an *algebraic open book decomposition* of X). In the case that X is unimodular (hence isomorphic to \mathbb{R}^3 or Sol_3), then item 6 of Theorem 3.6 in [21] implies that all 2-dimensional subgroups of X are minimal, hence $P_i^\pm(R)$ are minimal surfaces and the claim is proved in this case. Next we will prove the desired mean convexity of $U_i(R)$ in the case that X is non-unimodular and for $i = 1$ (for $i = 2$ the argument is similar and we leave it for the reader). Item 5 of Theorem 3.6 in [21] ensures that up to possibly rescaling the metric, X is isometric and isomorphic to $\mathbb{R}^2 \rtimes_{A(b)} \mathbb{R}$ for a diagonal matrix of the form $A(b) = \begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}$, for some $b \in \mathbb{R}, b \neq -1$; furthermore, we can assume that $P_1 = \{y = 0\}$ and thus, the algebraic open book decomposition of $\mathbb{R}^2 \rtimes_{A(b)} \mathbb{R}$ that contains P_1 as one of its leaves is $\mathcal{A}_1 = \{H_1(\lambda) \mid \lambda \in \mathbb{R} \cup \{\infty\}\}$, where

$$H_1(\lambda) = \begin{cases} \{(x, \frac{\lambda}{b}(e^{bz} - 1), z) \mid x, z \in \mathbb{R}\} & \text{if } b \neq 0, \lambda \in \mathbb{R}, \\ \{(x, \lambda z, z) \mid x, z \in \mathbb{R}\} & \text{if } b = 0, \lambda \in \mathbb{R}, \\ \mathbb{R}^2 \rtimes_{A(b)} \{0\} & \text{if } \lambda = \infty; \end{cases}$$

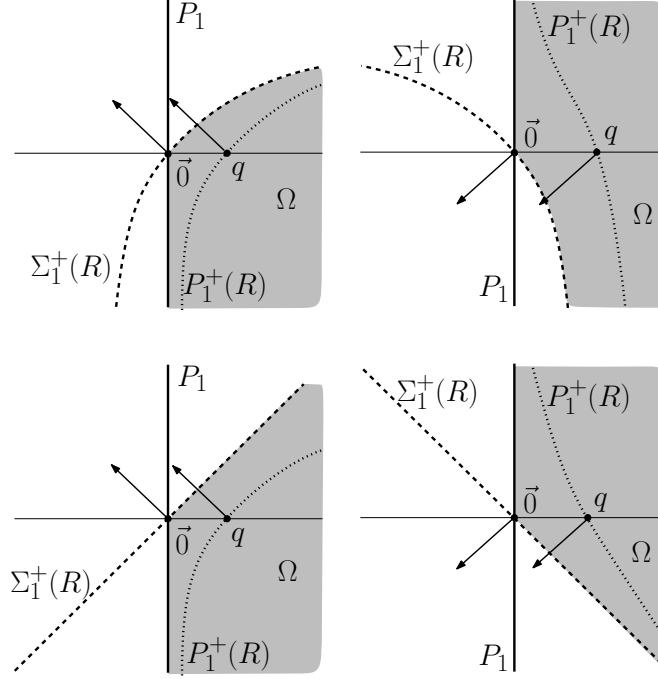


FIGURE 3. The mean curvature vector of $P_1^+(R) = q * \Sigma_1^+(R)$ (dotted) points towards the 2-dimensional subgroup $\Sigma_1^+(R)$ (dashed), and hence towards the vertical plane P_1 . Above: the case $b \neq 0$. Below: the case $b = 0$. All graphics are representations in the (y, z) -plane.

(hence $P_1 = H_1(0)$). Observe that the 2-dimensional subgroups in \mathcal{A}_1 are products with the x -factor of proper graphs of the z -variable in the (y, z) -plane; this applies in particular to P_1 and to $\Sigma_1^\pm(R)$. Therefore, ∂_x is everywhere tangent to P_1 and to $\Sigma_1^\pm(R)$. As $P_1^\pm(R)$ is a right coset of P_1 and $F_1 = \partial_x$ is a right invariant vector field, then ∂_x is also everywhere tangent to $P_1^\pm(R)$. In other words, $P_1^\pm(R)$ is the product with the x -factor of a curve in the (y, z) -plane. In fact, this curve must be a proper graph of the z -variable (to see this, observe that every horizontal plane $\mathbb{R}^2 \rtimes_{A(b)} \{z\}$ intersects $P_1^\pm(R)$ in a line parallel to the x -axis which is the set of points of $\mathbb{R}^2 \rtimes_{A(b)} \{z\}$ at distance R from P_1). Now the desired mean convexity of $U_1(R)$ with respect to the inward normal vector can be understood by considering the related problem for z -graphs in the (y, z) -plane (i.e., after taking quotients in the x -factor): to do this, observe that $P_1^+(R)$ lies entirely at one side of P_1 , say its right side, see Figure 3. We can write $P_1^+(R) = q * \Sigma_1^+(R)$, the left coset of $\Sigma_1^+(R)$ obtained after left multiplication by an element $q \in P_1^+(R)$. Observe that if $\Sigma_1^+(R) = P_1$ then $\Sigma_1^+(R)$ is minimal (see Remark 2.2), and hence $P_1^\pm(R)$ is minimal as well, which gives the desired mean convexity in this case. Thus, we can assume that $\Sigma_1^+(R) \neq P_1$. As $P_1^+(R)$ lies at the right side of P_1 and is disjoint from $\Sigma_1^+(R)$, then $P_1^+(R)$ lies in the component Ω of $[\mathbb{R}^2 \rtimes_{A(b)} \mathbb{R}] - [P_1 \cup \Sigma_1^+(R)]$ that contains $[\mathbb{R}^2 \rtimes_{A(b)} \{0\}] \cap \{y > 0\}$ (see Figure 3). As the mean curvature vector of $P_1^+(R)$ at q equals the mean curvature vector of $\Sigma_1^+(R)$ at $\vec{0}$, then a continuity argument in the variable R gives that the mean curvature vector of $P_1^+(R)$ at $q = q(R)$ points towards P_1 , which finishes the proof of the claim.

By the last claim, the maximum principle (for the case X is unimodular) and the mean curvature comparison principle (when X is non-unimodular) applied to the foliation $\{P_i^-(R) \mid R > 0\} \cup \{P_i\} \cup \{P_i^+(R) \mid R > 0\}$ gives that every compact minimal surface M with boundary in $\mathcal{W}(\Gamma, r)$ lies in the domain $U_i(r)$; hence, $M \subset U_1(r) \cap U_2(r)$. If we prove that $U_1(r) \cap U_2(r) \subset \mathcal{W}(\Gamma, 2r)$, then we would deduce that $M \subset \mathcal{W}(\Gamma, 2r)$, i.e. the proposition holds with $j = 2$ in this case of X admitting two distinct vertical planes which are subgroups of X . To check that $U_1(r) \cap U_2(r) \subset \mathcal{W}(\Gamma, 2r)$, let $p = (x, y, z) \in X$; we will denote by $p_1 = (x, 0, z) \in P_1$, $p_2 = (0, y, z) \in P_2$ and $p_3 = (0, 0, z) \in \Gamma$. Assume that the following properties concerning the extrinsic distance d in X hold (we will prove them later):

(A) $d(p, p_i) = d(p, P_i)$, for $i = 1, 2$.

(B) $d(p, p_3) = d(p, \Gamma)$.

Under these assumptions, we have:

$$d(p, \Gamma) = d(p, p_3) \leq d(p, p_1) + d(p_1, p_3) = d(p, P_1) + d(p_1, p_3) \stackrel{(\star)}{=} d(p, P_1) + d(p, p_2) = d(p, P_1) + d(p, P_2),$$

where in (\star) we have left multiplied by $(0, y, 0)$ (which is an ambient isometry of X , thus preserves distances) in the second summand. Now the inclusion $U_1(r) \cap U_2(r) \subset \mathcal{W}(\Gamma, 2r)$ follows directly. We next prove (A) and (B). Given $q \in X$ and $A \subset X$, let $\mathcal{C}(p, q)$ (resp. $\mathcal{C}(p, A)$) be the set of piecewise smooth curves $\alpha: [0, 1] \rightarrow X$ such that $\alpha(0) = p$ and $\alpha(1) = q$ (resp. $\alpha(1) \in A$). Consider the maps

$$\begin{aligned} \Theta_1: \mathcal{C}(p, P_1) &\rightarrow \mathcal{C}(p, p_1), & [\Theta_1(\alpha)](t) &= (x, y(t), z), \\ \Theta_2: \mathcal{C}(p, P_2) &\rightarrow \mathcal{C}(p, p_2), & [\Theta_2(\alpha)](t) &= (x(t), y, z), \\ \Theta_3: \mathcal{C}(p, \Gamma) &\rightarrow \mathcal{C}(p, p_3), & [\Theta_3(\alpha)](t) &= (x(t), y(t), z), \end{aligned}$$

if $\alpha(t) = (x(t), y(t), z(t))$, $t \in [0, 1]$. Since $e^{zA(b)} = \begin{pmatrix} e^z & 0 \\ 0 & e^{bz} \end{pmatrix}$, then equation (2.7) implies that Θ_i decreases lengths of curves, for each $i = 1, 2, 3$. Thus, for $i = 1, 2$ we have

$$d(p, P_i) = \inf_{\alpha \in \mathcal{C}(p, P_i)} \text{Length}(\alpha) \geq \inf_{\alpha \in \mathcal{C}(p, P_i)} \text{Length}(\Theta_i(\alpha)) \geq \inf_{\beta \in \mathcal{C}(p, p_i)} \text{Length}(\beta) = d(p, p_i).$$

Since $p_i \in P_i$, then the last inequality is in fact an equality and (A) is proved. To prove (B) one uses the same argument, changing P_i by Γ , Θ_i by Θ_3 and p_i by p_3 . This finishes the proof of the proposition when X is not isomorphic to $\tilde{E}(2)$ or Nil_3 .

Assume now that X is isomorphic to $\tilde{E}(2)$. Thus, after scaling the metric of X , it is isomorphic and isometric to $\mathbb{R}^2 \rtimes_{A(c)} \mathbb{R}$ with $A(c) = \begin{pmatrix} 0 & -c \\ 1/c & 0 \end{pmatrix}$ for some $c > 0$ (see Section 2.7 of [21]).

For $t \in \mathbb{R}$, define the vertical planes $P_1(t) = \{x = t\}$, $P_2(t) = \{y = t\}$. For any $t > 0$ and $i = 1, 2$, let $U_i(t)$ be the slab in X with boundary $P_i(-t) \cup P_i(t)$. Equation (2.1) gives that the left translation in X by an element of the form $(0, 0, m\pi)$ with $m \in 2\mathbb{Z}$ writes as $(\mathbf{0}, m\pi) * (\mathbf{p}_2, z_2) = (e^{m\pi A(c)} \mathbf{p}_2, m\pi + z_2) = (\mathbf{p}_2, m\pi + z_2) = (\mathbf{p}_2, z_2) * (\mathbf{0}, m\pi)$, for all $\mathbf{p}_2 \in \mathbb{R}^2$, $z_2 \in \mathbb{R}$. In particular, for all $t > 0$ the slab $U_i(t)$ is invariant under the isometry which is given by left (or right) translation by $(0, 0, m\pi)$; note that $\mathcal{W}(\Gamma, r)$ is also invariant under left translation by $(0, 0, m\pi)$, for all $r > 0$. Since for $r > 0$ fixed, the family of sets

$$\{U_1(nr) \cap U_2(nr) \cap \{|z| \leq \pi\} \mid n \in \mathbb{N}\}$$

forms a compact exhaustion for the horizontal slab $\{|z| \leq \pi\}$ and $\mathcal{W}(\Gamma, r) \cap \{|z| \leq \pi\}$ is compact, then there exists a $k \in \mathbb{N}$ such that $\mathcal{W}(\Gamma, r) \cap \{|z| \leq \pi\} \subset U_1(kr) \cap U_2(kr) \cap \{|z| \leq \pi\}$.

Furthermore, this integer k can be chosen independently from r (because the identity map from $\{|z| \leq \pi\} = \mathbb{R}^2 \rtimes_A [-\pi, \pi]$ into $\mathbb{R}^2 \times [-\pi, \pi]$ with the product metric is a quasi-isometry, for every $A \in \mathcal{M}_2(\mathbb{R})$). The left-invariance property of $U_i(kr)$ and $\mathcal{W}(\Gamma, r)$ by left translation by $(0, 0, 2\pi)$ implies that $\mathcal{W}(\Gamma, r)$ is contained in $U_1(kr) \cap U_2(kr)$. This implies that if $M \subset X$ is a compact minimal surface with $\partial M \subset \mathcal{W}(\Gamma, r)$, then $\partial M \subset U_1(kr) \cap U_2(kr)$, and by the maximum principle applied to the family of minimal surfaces $\{P_i(t) \mid t \in \mathbb{R}\}$, $i = 1, 2$, we deduce that $M \subset U_1(kr) \cap U_2(kr)$. On the other hand, since the sets $\mathcal{W}(\Gamma, r) \cap \{|z| \leq \pi\}$ also form a compact exhaustion for the slab $\{|z| \leq \pi\}$, then similar reasoning shows that there exists a $j \in \mathbb{N}$ independent of r such that $U_1(kr) \cap U_2(kr) \subset \mathcal{W}(\Gamma, jr)$, from where we conclude that $M \subset \mathcal{W}(\Gamma, jr)$. This finishes the proof of the proposition in the case X is isomorphic to $\tilde{E}(2)$.

Suppose now that X is isomorphic to Nil_3 . After a scaling of the metric, we may assume that X is $\mathbb{R}^2 \rtimes_A \mathbb{R}$ with $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. In this case there exists a unique vertical plane which is a subgroup of X , namely $P_1 = \{y = 0\}$. Since Nil_3 is unimodular, Theorem 3.6 in [21] implies that the foliation of surfaces at constant distance from P_1 consists of minimal surfaces in X . Given $r > 0$, let $P_1^\pm(r) = \{y = \pm r\}$ be the boundary planes of the closed regular neighborhood $U_1(r)$ of P_1 of radius r (one can check that the distance from $\vec{0}$ to $(0, \pm r, 0)$ is r), and let $S^\pm(r, R) \subset P_1^\pm(r)$ be the round circle of Euclidean radius $R > 0$ centered at the point $(0, \pm r, 0)$. **We claim that for R much larger than r , there exists an embedded minimal annulus $A(r, R) \subset X$ with boundary $\partial A(r, R) = S^+(r, R) \cup S^-(r, R)$.** To see this, first note that

- (I) $U_1(r)$ is quasi-isometric to the Riemannian product $\mathbb{R}^2 \times [-r, r]$ under the mapping arising from normal coordinates on P_1 . This is because, in the fixed compact regular neighborhood of radius 1 of the segment $\{(0, t, 0) \mid t \in [-r, r]\}$ in the slab $U_1(r)$, the restricted mapping is a quasi-isometry with its image and the differential of the normal coordinate map is invariant under left translations by elements in P_1 .
- (II) The Euclidean area of the cylinder $C(r, R) = \{(x, y, z) \mid (x, r, z) \in S^+(r, R), |y| \leq r\}$ is $4\pi r R$.

From (I), (II) we deduce that the area in X of $C(r, R)$ is less than $4\pi c r R$, for some $c > 0$ that only depends on r . As the union of the two disks $D^\pm(r, R) \subset P_1^\pm(r)$ bounded by $S^\pm(r, R)$ has area $2\pi R^2$ and each of these disks is area-minimizing in X (in fact, $D^\pm(r, R)$ is the unique solution of the Plateau problem for boundary $S^\pm(r, R)$ in X), then the Douglas Criterion and the Geometric Dehn's Lemma for Planar Domains in Theorem 5 in [22] (as adapted in the more general boundary setting of [23]) guarantee that for $R \gg r$, there exists an embedded least-area annulus $A(r, R)$ in X with boundary $\partial A(r, R) = S^+(r, R) \cup S^-(r, R)$, and our claim is proved.

Consider the family of minimal annuli $\mathcal{F} = \{p * A(r, R) \mid p \in P_1, [p * A(r, R)] \cap \mathcal{W}(\Gamma, r) = \emptyset\}$, where $p * A(r, R)$ denotes the left translation of $A(r, R)$ by the element p . Observe that \mathcal{F} satisfies the following properties.

- (\mathcal{F} -I) \mathcal{F} is nonempty. Furthermore, \mathcal{F} is invariant under left translation by elements of Γ and under the rotation R_Γ by $\pi/2$ around Γ (this follows from the invariance of $\mathcal{W}(\Gamma, r)$ and of P_1 under these ambient isometries of X).
- (\mathcal{F} -II) There exists $k > r$ such that for all $t \in [k, \infty)$, we have

$$[(t, 0, 0) * (A(r, R))] \cap \mathcal{W}(\Gamma, r) = \emptyset \text{ and } [(-t, 0, 0) * R_\Gamma(A(r, R))] \cap \mathcal{W}(\Gamma, r) = \emptyset.$$

Let \mathcal{F}' be the family of left translates of $A(r, R)$ and of $R_\Gamma(A(r, R))$ appearing in Property (\mathcal{F} -II) together with their left translates by elements in Γ . It follows that there exists a $j \in \mathbb{N}$ such that

$$(5.1) \quad U_1(r) - (\cup_{F \in \mathcal{F}'} F) \subset \mathcal{W}(\Gamma, jr);$$

the existence of j follows from similar arguments as those in the proof of Property (I) above.

Finally, consider a compact minimal surface $M \subset X$ with boundary $\partial M \subset \mathcal{W}(\Gamma, r)$. Since $\mathcal{W}(\Gamma, r) \subset U_1(r)$ and the boundary planes of $U_1(r)$ are minimal, then the maximum principle implies that $M \subset U_1(r)$. To finish the proof of Proposition 5.1, we will check that $M \subset \mathcal{W}(\Gamma, jr)$. Otherwise, as $M \subset U_1(r)$ then (5.1) implies that M intersects some annulus $F \in \mathcal{F}'$. But then by compactness of M and F , there exists a largest $t > 0$ such that $M \cap [(t, 0, 0) * F] \neq \emptyset$, which gives a contradiction to the maximum principle since the boundary of $(t, 0, 0) * F$ is disjoint from ∂M . Now the proof of Proposition 5.1 is complete. \square

THEOREM 5.2. *Let $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ be a metric semidirect product, and let $\Gamma \subset X$ be a vertical geodesic. Then, given $r, C > 0$ there exists $R = R(r, C) > 0$ such that for every compact immersed minimal surface $M \subset \mathcal{W}(\Gamma, r)$ with the norm of its second fundamental form less than C , the radius of M is less than R .*

PROOF. After a fixed left translation of Γ , we will assume that Γ is the z -axis in $\mathbb{R}^2 \rtimes_A \mathbb{R}$.

To prove the theorem we proceed by contradiction. Suppose that there exist $r, C > 0$ and a sequence of compact, immersed minimal surfaces $h_n: M_n \looparrowright \mathcal{W}(\Gamma, r)$ with the norm of their second fundamental forms less than C and such that there exist points $p_n \in M_n$ for which the intrinsic distances from p_n to the boundaries of the M_n satisfy $d_{M_n}(p_n, \partial M_n) > n$ for all $n \in \mathbb{N}$. Consider the compact domain $Y = \mathcal{W}(\Gamma, r) \cap [\mathbb{R}^2 \rtimes_A \{0\}]$. After left translating the immersions h_n appropriately by elements in the 1-parameter subgroup $\Gamma = \{(0, 0, s) \in \mathbb{R}^2 \rtimes_A \mathbb{R} \mid s \in \mathbb{R}\}$ and passing to a subsequence, we may assume that $h_n(p_n) \in Y$ for all n and this sequence of points converges to a point $q_\infty \in Y$.

Since the minimal immersions h_n have uniform curvature estimates, then there exists a complete, connected immersed minimal surface $h_\infty: M_\infty \looparrowright \mathcal{W}(\Gamma, r)$ of bounded second fundamental form that is a limit of the restriction of (a subsequence, denoted in the same way, of) the h_n to certain smooth compact domains $\Omega_n \subset M_n$ with $p_n \in \Omega_n$ and $d_n(p_n, \partial \Omega_n) > n$ for all $n \in \mathbb{N}$, and such that $h_\infty(p_\infty) = q_\infty$ for some point $p_\infty \in M_\infty$. Now consider the closure \mathcal{M} of the union of all left translations of $h_\infty(M_\infty)$ by elements in Γ , i.e.,

$$\mathcal{M} = \overline{\{a * h_\infty(M_\infty) \mid a \in \Gamma\}},$$

which is a connected subset of $\mathcal{W}(\Gamma, r)$. As $h_\infty(M_\infty)$ has bounded second fundamental form, then, by the same compactness arguments, given any point $q \in \mathcal{M}$, there exists a compact embedded minimal disk $D(q) \subset \mathcal{M}$ with $q \in \text{Int}(D(q))$.

We next consider the special case where X is non-unimodular, and so we assume $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ with A satisfying (2.9); see Remark 2.4. Let L be the line given by Corollary 4.3. For $p = (x, y, 0) \in L$ with $x^2 + y^2$ sufficiently large, the set Y lies in the interior of the strip $S \subset \mathbb{R}^2 \rtimes_A \{0\}$ bounded by the pair of Euclidean lines $L_p, L_{-p} \subset \mathbb{R}^2 \rtimes_A \{0\}$ that are orthogonal to L at the respective points $p, -p$. By Corollary 4.3, there exist circles $C_p, C_{q=-p}$ with $p \in C_p, q \in C_q$ such that $C_p \cup C_q$ does not intersect the interior of the strip S and $C_p \cup C_q$ is the boundary of a least-area embedded annulus $\Sigma \subset \mathbb{R}^2 \rtimes_A [0, \infty)$. Let L^\perp in $\mathbb{R}^2 \rtimes_A \{0\}$ be the line perpendicular to L at $\vec{0}$. For $t \in L^\perp$, let $t * \Sigma$ denote the left translation of Σ by t . Note that for some $t_0 \in \mathbb{R}$, $(t_0 * \Sigma) \cap \mathcal{M} \neq \emptyset$ and that for $|t|$ large, $(t * \Sigma) \cap \mathcal{M} = \emptyset$. It follows that there exists a $t_1 \in L^\perp$ with largest norm such

that $t_1 * \Sigma$ intersects \mathcal{M} at some point p_{t_1} and near p_{t_1} the set \mathcal{M} lies on one side of $t_1 * \Sigma$. This implies that there exists an embedded minimal disk $D(p_{t_1}) \subset \mathcal{M}$ containing p_{t_1} , such that $D(p_{t_1})$ lies on one side of $t_1 * \Sigma$. Now the maximum principle for minimal surfaces gives that $t_1 * \Sigma \subset \mathcal{M}$, which is false since $\partial(t_1 * \Sigma) \cap \mathcal{M} = \emptyset$. This contradiction proves the proposition in this special case where X is non-unimodular.

Finally, we consider the remaining case where X is unimodular (hence $\text{tr}(A) = 0$). Consider the foliation of X by minimal planes produced in the proof of Proposition 5.1, i.e. (with the notation in that proposition), the planes $P_1^-(R)$ at constant distance $R > 0$ from P_1 in the case that X is isomorphic to Sol_3 , Nil_3 or \mathbb{R}^3 , and the periodic planes $P_1(t) = \{x = t\}$, in the case of X is isomorphic to $\tilde{\text{E}}(2)$. Since in all of these cases there exist one of these minimal planes P such that $P \cap \mathcal{M} \neq \emptyset$ and \mathcal{M} lies on one side of P , the argument in the previous paragraph with P in place of Σ easily generalizes to give a contradiction. This contradiction completes the proof of Theorem 5.2. \square

As a direct consequence of Theorem 5.2 and the classical curvature estimates for stable minimal surfaces [32, 34], we obtain:

COROLLARY 5.3. *Let $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ be a metric semidirect product, and let $\mathcal{W}(\Gamma, r) \subset X$ denote a solid metric cylinder in X of radius $r > 0$ around a vertical geodesic $\Gamma \subset X$. Then, there are no complete stable minimal surfaces contained in $\mathcal{W}(\Gamma, r)$.*

In order to prove Theorem 1.1 stated in the Introduction we will need an auxiliary construction for the case that X is non-unimodular with positive Milnor D -invariant. To do this, in the remainder of this section we fix $\alpha \in [0, 1)$ and $\beta \in [0, \infty)$, we consider the matrix $A = A(\alpha, \beta)$ given by (2.9), and the non-unimodular metric Lie group $X = X(\alpha, \beta) = \mathbb{R}^2 \rtimes_A \mathbb{R}$ with its usual left invariant metric $\langle \cdot, \cdot \rangle$ determined by A (see Definition 2.1). Under our hypotheses on α, β , we have that the Milnor D -invariant of X is positive. Conversely, every non-unimodular metric Lie group with positive Milnor D -invariant can be expressed as $X(\alpha, \beta)$ for some $\alpha \in [0, 1)$ and $\beta \geq 0$, see Section 2.

The 1-parameter subgroup $\{(0, 0, s) \in \mathbb{R}^2 \rtimes_A \mathbb{R} \mid s \in \mathbb{R}\}$ of X generates under left multiplication a right invariant vector field F_3 of X (see equation (2.3) where the notation for the matrix A is different from the one used here). Next we will study the mean convexity of solid cylinders in X obtained after flowing the domains enclosed by a family of homothetic ellipses in $\mathbb{R}^2 \rtimes_A \{0\}$ through the 1-parameter group of isometries $\{\phi_s \mid s \in \mathbb{R}\}$ associated to F_3 , namely the left translations by elements in the z -axis of $\mathbb{R}^2 \rtimes_A \mathbb{R}$. The technical property stated in Proposition 5.4 will be used in Theorem 5.6 below, in order to obtain the desired radius estimate for stable minimal surfaces in X that generalizes Corollary 5.3.

Consider an ellipse $C_\mu = \{(x, y, 0) \in \mathbb{R}^2 \rtimes_A \{0\} \mid x^2 + \frac{y^2}{\mu^2} = 1\}$, where $\mu > 0$ is to be determined, and the family of homothetic ellipses

$$(5.2) \quad rC_\mu = \{(rx, ry, 0) \mid (x, y, 0) \in C_\mu\}, \quad r > 0.$$

Let $rE_\mu \subset \mathbb{R}^2 \rtimes_A \{0\}$ denote the compact disk with boundary rC_μ , and let

$$(5.3) \quad \Omega(r) = \bigcup_{s \in \mathbb{R}} \phi_s(rE_\mu)$$

be the F_3 -invariant closed solid cylinder obtained after flowing rE_μ by the isometries that generate F_3 .

PROPOSITION 5.4. *Let $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ be a metric semidirect product where A is as in equation (2.9) with $\alpha \in [0, 1)$ and $\beta \in [0, \infty)$. Then, there exist $\mu > 0$ and $r_0 > 0$ such that the F_3 -invariant solid cylinder $\Omega(r)$ over rE_μ defined in (5.3) is strictly mean convex for every $r \geq r_0$.*

PROOF. Fix a positive μ to be determined later. Given $r > 0$, parameterize rC_μ by $\gamma = \gamma(t) = (x(t), y(t), 0)$, where

$$(5.4) \quad x(t) = r \cos t, \quad y(t) = r\mu \sin t, \quad t \in [0, 2\pi].$$

A parametrization Φ of the F_3 -invariant cylinder given by the boundary $\Sigma = \Sigma(r)$ of $\Omega(r)$ is obtained by flowing γ through the 1-parameter group $\{\phi_s \mid s \in \mathbb{R}\}$, i.e.,

$$\Phi(t, s) = \phi_s(\gamma(t)), \quad (t, s) \in [0, 2\pi] \times \mathbb{R},$$

where $\phi_s(\mathbf{p}, z) = (e^{sA}\mathbf{p}, s+z)$ for all $s, z \in \mathbb{R}$ and $\mathbf{p} \in \mathbb{R}^2$ (\mathbf{p} is considered to be a column vector).

The mean curvature $H = H(t, s)$ of Σ is given by the well-known formula

$$(5.5) \quad 2(EG - F^2)H = eG - 2fF + gE,$$

where E, F, G and e, f, g are respectively the coefficients of the first and second fundamental form of Σ (these coefficients are functions of (t, s)):

$$(5.6) \quad \begin{aligned} E &= \|\Phi_t\|^2, & F &= \langle \Phi_t, \Phi_s \rangle, & G &= \|\Phi_s\|^2, \\ e &= \langle N, \nabla_{\Phi_t} \Phi_t \rangle, & f &= \langle N, \nabla_{\Phi_t} \Phi_s \rangle, & g &= \langle N, \nabla_{\Phi_s} \Phi_s \rangle, \end{aligned}$$

where $\Phi_t = \frac{\partial \Phi}{\partial t}$, $\Phi_s = \frac{\partial \Phi}{\partial s}$ and $N = \frac{\Phi_t \times \Phi_s}{\|\Phi_t \times \Phi_s\|}$ is the unit normal vector field to Σ . Observe that $\Phi_t(t, 0)$ defines the counterclockwise orientation on rC_μ and that $\Phi_s(t, 0)$ points upward. Therefore, $N(t, 0)$ points outward $\Omega(r)$ along γ . Since Σ is F_3 -invariant, the strict mean convexity of $\Omega(r)$ will follow from the existence of some $\mu > 0$ (depending solely on α, β) such that the function $t \in [0, 2\pi] \mapsto (eG - 2fF + gE)(t, 0)$ is strictly negative for $r > 0$ large enough.

Note that

$$(5.7) \quad \Phi_t(t, 0) = \gamma'(t) = \begin{pmatrix} x'(t) \\ y'(t) \\ 0 \end{pmatrix} = \begin{bmatrix} x'(t) \\ y'(t) \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\mu}y(t) \\ \mu x(t) \\ 0 \end{bmatrix},$$

where the parentheses (resp. brackets) refer to coordinates with respect to the basis $\{\partial_x, \partial_y, \partial_z\}$ (resp. to the usual orthonormal basis $\{E_1, E_2, E_3\}$ of the Lie algebra of X given by (2.4)); in general, the change of coordinates between the two bases at a point $(x, y, z) \in \mathbb{R}^2 \rtimes_A \mathbb{R}$ is

$$(5.8) \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{pmatrix} e^{zA} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \end{pmatrix}, \quad a, b, c \in \mathbb{R}.$$

Also, $\Phi_s(t, s) = (F_3)_{\Phi(t, s)}$ and so, the globally defined right invariant vector field F_3 extends Φ_s . Using (2.3) (recall that the entries of the matrix A are given by (2.9)), we have

$$(5.9) \quad F_3(x, y, z) = \begin{pmatrix} \delta \\ \varepsilon \\ 1 \end{pmatrix} \stackrel{(5.8)}{=} \begin{bmatrix} \delta a_{11}(-z) + \varepsilon a_{12}(-z) \\ \delta a_{21}(-z) + \varepsilon a_{22}(-z) \\ 1 \end{bmatrix},$$

where

$$(5.10) \quad \delta(x, y) = (1 + \alpha)x - (1 - \alpha)\beta y, \quad \varepsilon(x, y) = (1 + \alpha)\beta x + (1 - \alpha)y,$$

and $a_{ij}(z)$ are the entries of the matrix e^{zA} , see (2.5). In particular,

$$(5.11) \quad \Phi_s(t, 0) = (F_3)_{\gamma(t)} = \begin{bmatrix} \delta(t) \\ \varepsilon(t) \\ 1 \end{bmatrix},$$

where $\delta(t) = \delta(\gamma(t))$ and $\varepsilon(t) = \varepsilon(\gamma(t))$.

From (5.7), (5.11) we can compute the coefficients of the first fundamental form at points of the form $\Phi(t, 0)$:

$$(5.12) \quad \begin{cases} E(t, 0) &= x'(t)^2 + y'(t)^2, \\ F(t, 0) &= \delta(t)x'(t) + \varepsilon(t)y'(t), \\ G(t, 0) &= 1 + \delta(t)^2 + \varepsilon(t)^2. \end{cases}$$

The unit normal vector field at points of the form $\Phi(t, 0)$ is given by

$$(5.13) \quad N(t, 0) = \frac{1}{\Delta(t)} (\Phi_t \times \Phi_s)(t, 0) = \frac{1}{\Delta(t)} \begin{bmatrix} y'(t) \\ -x'(t) \\ \varepsilon(t)x'(t) - \delta(t)y'(t) \end{bmatrix},$$

where $\Delta(t) = \|\Phi_t \times \Phi_s\|(t, 0)$.

We next compute the coefficients of the second fundamental form of Σ . Using (5.7) and denoting by $\frac{DW}{dt}$ the covariant derivative of a vector field W along γ , we have

$$(5.14) \quad \begin{aligned} (\nabla_{\Phi_t} \Phi_t)(t, 0) &\stackrel{(5.7)}{=} \frac{D}{dt} (x'(t)(E_1)_{\gamma(t)} + y'(t)(E_2)_{\gamma(t)}) \\ &= \begin{bmatrix} x''(t) \\ y''(t) \\ 0 \end{bmatrix} + x'(t)\nabla_{\gamma'(t)} E_1 + y'(t)\nabla_{\gamma'(t)} E_2. \\ &\stackrel{(2.8)}{=} \begin{bmatrix} x''(t) \\ y''(t) \\ (1 + \alpha)x'(t)^2 + 2\alpha\beta x'(t)y'(t) + (1 - \alpha)y'(t)^2 \end{bmatrix}. \end{aligned}$$

Analogously,

$$(5.15) \quad \begin{aligned} (\nabla_{\Phi_t} \Phi_s)(t, 0) &= \frac{D(F_3 \circ \gamma)}{dt} \stackrel{(5.11)}{=} \frac{D}{dt} (\delta(t)(E_1)_{\gamma(t)} + \varepsilon(t)(E_2)_{\gamma(t)} + (E_3)_{\gamma(t)}) \\ &= \begin{bmatrix} \delta'(t) \\ \varepsilon'(t) \\ 0 \end{bmatrix} + \delta(t)\nabla_{\gamma'(t)} E_1 + \varepsilon(t)\nabla_{\gamma'(t)} E_2 + \nabla_{\gamma'(t)} E_3 \\ &\stackrel{(2.8)}{=} \begin{bmatrix} \delta'(t) - (1 + \alpha)x'(t) - \alpha\beta y'(t) \\ \varepsilon'(t) - \alpha\beta x'(t) - (1 - \alpha)y'(t) \\ \delta(t)[(1 + \alpha)x'(t) + \alpha\beta y'(t)] + \varepsilon(t)[\alpha\beta x'(t) + (1 - \alpha)y'(t)] \end{bmatrix}. \end{aligned}$$

To compute $g = \langle N, \nabla_{\Phi_s} \Phi_s \rangle = \langle N, \nabla_{\Phi_s} F_3 \rangle$ we use that F_3 is a Killing vector field that extends Φ_s :

$$\begin{aligned} g(t, s) &= -\langle \Phi_s, \nabla_N F_3 \rangle = -\frac{1}{2} N(\|F_3\|^2) \\ &\stackrel{(5.9)}{=} -(\delta a_{11}(-z) + \varepsilon a_{12}(-z)) N(\delta a_{11}(-z) + \varepsilon a_{12}(-z)) \end{aligned}$$

$$-(\delta a_{21}(-z) + \varepsilon a_{22}(-z)) N(\delta a_{21}(-z) + \varepsilon a_{22}(-z)).$$

Hence,

$$(5.16) \quad g(t, 0) = -\delta(t) \{N(\delta) + \delta(t)N(a_{11}(-z)) + \varepsilon(t)N(a_{12}(-z))\} \\ -\varepsilon(t) \{N(\varepsilon) + \delta(t)N(a_{21}(-z)) + \varepsilon(t)N(a_{22}(-z))\},$$

where we have simplified the notation $N(t, 0)$ by N . By using (5.10) and (5.13) one has (at $(t, 0)$):

$$(5.17) \quad \begin{cases} N(\delta) &= \frac{1}{\Delta(t)} [(1 + \alpha)y'(t) + (1 - \alpha)\beta x'(t)], \\ N(\varepsilon) &= \frac{1}{\Delta(t)} [(1 + \alpha)\beta y'(t) - (1 - \alpha)x'(t)], \\ N(a_{ij}(-z)) &= -\frac{1}{\Delta(t)} [\varepsilon(t)x'(t) - \delta(t)y'(t)] a'_{ij}(0), \quad \text{for } i = 1, 2. \end{cases}$$

Since $(a_{ij}(z))_{i,j} = e^{zA}$, then $(a'_{ij}(0))_{i,j} = A$. Now, (5.16) and (5.17) give

$$(5.18) \quad \Delta(t)g(t, 0) = -\{(1 + \alpha)y' + (1 - \alpha)\beta x' - [(1 + \alpha)\delta - (1 - \alpha)\beta\varepsilon](\varepsilon x' - \delta y')\} \delta \\ -\{(1 + \alpha)\beta y' - (1 - \alpha)x' - [(1 + \alpha)\beta\delta + (1 - \alpha)\varepsilon](\varepsilon x' - \delta y')\} \varepsilon.$$

A direct substitution from (5.4), (5.6), (5.10), (5.14), (5.15) and (5.18) gives that

$$\Delta(t)(eG - 2fF + gE)(t, 0) = -\mu r^2 + r^4 h_{\alpha, \beta, \mu}(t) \\ -\frac{r^6}{4} \{2\mu + 2\alpha\mu \cos(2t) + \beta[1 + \alpha - (1 - \alpha)\mu^2] \sin(2t)\}^3,$$

where $h_{\alpha, \beta, \mu}(t)$ is a smooth, π -periodic function of t depending on the parameters α, β, μ . From the last displayed expression we deduce that the mean curvature of Σ with respect to N is strictly negative for all r large enough provided that the expression

$$\varrho_{\alpha, \beta, \mu}(t) = 2\mu + 2\alpha\mu \cos(2t) + \beta[1 + \alpha - (1 - \alpha)\mu^2] \sin(2t)$$

is positive as a function of $t \in [0, 2\pi]$, for any given values $\alpha \in [0, 1)$, $\beta \in [0, \infty)$ and for some choice of $\mu = \mu(\alpha, \beta) > 0$. Clearly,

$$\varrho_{\alpha, \beta, \mu}(t) = 2\mu + \langle u, v(t) \rangle \geq 2\mu - \|u\|,$$

where $u = u(\alpha, \beta, \mu) = (2\alpha\mu, \beta[1 + \alpha - (1 - \alpha)\mu^2])$, $v(t) = (\cos(2t), \sin(2t)) \in \mathbb{R}^2$ and both the last inner product and norm refer to the usual flat metric in \mathbb{R}^2 . Therefore, the proposition will be proved if we show that the following elementary property holds:

(R) Given $(\alpha, \beta) \in [0, 1) \times [0, \infty)$, there exists $\mu > 0$ such that

$$4\mu^2 > \|u\|^2 = 4\alpha^2\mu^2 + \beta^2[1 + \alpha - (1 - \alpha)\mu^2]^2.$$

If $\beta = 0$, then Property (R) clearly holds as $\alpha^2 < 1$. If $\beta > 0$, then the proof of Property (R) follows from an elementary analysis of the function $\chi(\lambda) = 4(1 - \alpha^2)\lambda - \beta^2[1 + \alpha - (1 - \alpha)\lambda]^2$, which has a (unique) maximum at $\lambda_0 = \frac{1+\alpha}{1-\alpha} (1 + 2\beta^{-2}) > 0$, with value $\chi(\lambda_0) = 4(1 + \alpha)^2 (1 + \beta^{-2}) > 0$. Now the desired $\mu > 0$ can be chosen as $\mu = \sqrt{\lambda_0}$. This completes the proof of the proposition. \square

As a consequence of the mean convexity of $\Omega(r)$, we have:

PROPOSITION 5.5. *Let $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ be a non-unimodular metric semidirect product, where A is as in equation (2.9) with $\alpha \in [0, 1)$ and $\beta \in [0, \infty)$. Let $\mu, r_0 > 0$ and $\Omega(r)$ be the numbers and related F_3 -invariant, mean convex solid cylinder given in Proposition 5.4. Suppose that $r > r_0$. Then every compact immersed minimal surface $M \subset X$ whose boundary lies in $\Omega(r)$ satisfies that $M \subset \Omega(r)$.*

PROOF. It is a consequence of a standard mean curvature comparison argument based on the following two facts:

- For every $r \geq r_0$, $\Omega(r)$ has mean convex boundary by Proposition 5.4, and $\Omega(r_0) \subset \Omega(r)$.
- The collection of boundaries $\{\partial\Omega(r) \mid r \geq r_0\}$ forms a codimension-one foliation of $X - \Omega(r_0)$.

□

Finally, from Proposition 5.1, Theorem 5.2 and Proposition 5.5, we can conclude the desired radius estimate for compact stable minimal surfaces in metric semidirect products stated in Theorem 1.1:

THEOREM 5.6. *Let $X = \mathbb{R}^2 \rtimes_A \mathbb{R}$ be a metric semidirect product. Given $r > 0$ and any vertical geodesic $\Gamma \subset X$, there exists a positive number $\Lambda(r) > 0$ such that the following property holds: for any compact stable minimal surface M in X such that all points of its boundary ∂M are at distance at most r from Γ , the radius of M is at most $\Lambda(r)$.*

PROOF. We first consider the case where A is as in equation (2.9) with $\alpha \in [0, 1)$ and $\beta \in [0, \infty)$. Let M be a compact stable minimal surface in X whose boundary lies in the closed solid metric cylinder $\mathcal{W}(\Gamma, r)$ of radius $r > 0$ in X around a vertical geodesic Γ . Note that there exists some $r' = r'(r) > 0$ such that $\mathcal{W}(\Gamma, r) \subset \Omega(r')$, where $\Omega(r')$ is the mean convex solid cylinder given in Proposition 5.4. Then, by Proposition 5.5 we deduce that $M \subset \Omega(r')$. As there exists some $r'' = r''(r) > 0$ such that $\Omega(r') \subset \mathcal{W}(\Gamma, r'')$, we obtain from Theorem 5.2 and the Schoen-Ros curvature estimates for stable minimal surfaces [34, 32] the desired radius estimate.

If X is non-unimodular with A given by (2.9) and non-positive Milnor D -invariant, or else $\text{trace}(A) = 0$, then we apply Proposition 5.1 to conclude that every compact immersed stable minimal surface in X is contained in some metric cylinder $\mathcal{W}(\Gamma, r')$, where r' depends only on X and r . Then, as in the previous paragraph, Theorem 5.2 implies that M has a radius estimate that only depends on X and r . This last observation completes the proof. □

6. Appendix.

PROPOSITION 6.1. *Let X be a non-unimodular semidirect product $\mathbb{R}^2 \rtimes_A \mathbb{R}$ endowed with its canonical metric, where $A \in \mathcal{M}_2(\mathbb{R})$ is given by (2.9) for some constants $\alpha, \beta \geq 0$. Let $D = \det(A) = (1 - \alpha^2)(1 + \beta^2)$ be the Milnor D -invariant associated to the Lie group $\mathbb{R}^2 \rtimes_A \mathbb{R}$. Then, the following properties hold:*

1. *Given $z \in \mathbb{R}$, the exponential of the matrix zA is equal to*

$$(6.1) \quad e^{zA} = e^z [\mathbf{C}_D(z) I_2 + \mathbf{S}_D(z)(A - I_2)],$$

where $I_2 \in \mathcal{M}_2(\mathbb{R})$ is the identity matrix and

$$(6.2) \quad \mathbf{C}_D(t) = \begin{cases} \cosh(\sqrt{1-D}t) & \text{if } D < 1, \\ 1 & \text{if } D = 1, \\ \cos(\sqrt{D-1}t) & \text{if } D > 1, \end{cases} \quad \mathbf{S}_D(t) = \begin{cases} \frac{1}{\sqrt{1-D}} \sinh(\sqrt{1-D}t) & \text{if } D < 1, \\ t & \text{if } D = 1, \\ \frac{1}{\sqrt{D-1}} \sin(\sqrt{D-1}t) & \text{if } D > 1. \end{cases}$$

2. *The norms of ∂_x, ∂_y and their inner product with respect to the canonical metric are*

$$\begin{aligned} \|\partial_x\|^2 &= e^{-2z} \{ \beta^2(1 + \alpha)^2 \mathbf{S}_D(z)^2 + [\mathbf{C}_D(z) - \alpha \mathbf{S}_D(z)]^2 \} \\ \|\partial_y\|^2 &= e^{-2z} \{ \beta^2(1 - \alpha)^2 \mathbf{S}_D(z)^2 + [\mathbf{C}_D(z) + \alpha \mathbf{S}_D(z)]^2 \} \\ \langle \partial_x, \partial_y \rangle &= -2\alpha\beta e^{-2z} \mathbf{S}_D(z) [\mathbf{S}_D(z) + \mathbf{C}_D(z)]. \end{aligned}$$

In particular:

- 2a. if $D > 0$ then $\|\partial_x\|^2, \|\partial_y\|^2, \langle \partial_x, \partial_y \rangle$ decay exponentially as $z \rightarrow +\infty$, and so, the norm of every horizontal right invariant vector field in X decays exponentially as $z \rightarrow +\infty$ as well.
- 2b. If $D < 1$, then A is diagonalizable with distinct eigenvalues $\lambda_{\pm} = 1 \pm \sqrt{1 - D}$. Let $v_+, v_- \in \mathbb{R}^2$ be unitary eigenvectors of A associated to λ_+, λ_- , and let V_+, V_- be the horizontal right invariant vector fields in X determined by $V_{\pm}(0) = v_{\pm}$. Then, $\|V_{\pm}\|(x, y, z) = e^{-\lambda_{\pm}z}$ for all $(x, y, z) \in X$.

PROOF. Since $\text{trace}(A) = 2$ and $\det(A) = D$, then the characteristic equation for A gives $A^2 - 2A + DI_2 = 0$. From here it is straightforward to show that if we define $f: \mathbb{R} \rightarrow \mathcal{M}_2(\mathbb{R})$ by $f(z) = e^z [\mathbf{C}_D(z)I_2 + \mathbf{S}_D(z)(A - I_2)]$, then $f'(z) = Af(z)$ and $f(0) = I_2$ (for this, use that $\mathbf{S}'_D = \mathbf{C}_D$ and that $\mathbf{C}'_D = (1 - D)\mathbf{S}_D$), which gives item 1 of the proposition.

The three displayed equalities in item 2 of the proposition are also direct computations that only use (6.1) and the expression (2.7) of the canonical metric in terms of x, y, z . If $D > 0$, then (6.2) and the three displayed equalities in item 2 imply that $\|\partial_x\|^2, \|\partial_y\|^2, \langle \partial_x, \partial_y \rangle$ decay exponentially as $z \rightarrow +\infty$. If $D < 1$, then the characteristic equation of A has two distinct real roots $\lambda_{\pm} = 1 \pm \sqrt{1 - D}$, which implies that A is diagonalizable. After a fixed rotation in the (x, y) -plane around the origin (this change of coordinates does not affect either the Lie group structure in $\mathbb{R}^2 \rtimes_A \mathbb{R}$ or its canonical metric), we can assume that $V_+ = \partial_x$, i.e.,

$$A = \begin{pmatrix} \lambda_+ & b \\ 0 & \lambda_- \end{pmatrix} \text{ and thus, } e^{zA} = \begin{pmatrix} e^{\lambda_+z} & a_{12}(z) \\ 0 & e^{\lambda_-z} \end{pmatrix}$$

for certain $b \in \mathbb{R}$ and $a_{12}(z)$ function of z . Thus, (2.7) directly gives that $\|V_+\|(x, y, z) = \|\partial_x\|(x, y, z) = e^{-\lambda_+z}$. The proof of $\|V_-\|(x, y, z) = e^{-\lambda_-z}$ is analogous. \square

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